JOURNAL OF PURE MATHEMATICS

Volume 19, 2002

DEPARTMENT OF PURE MATHEMATICS, UNIVERSITY OF CALCUTTA

35, Ballygunge Circular Road, Kolkata - 700 019, India

JOURNAL OF PURE MATHEMATICS
UNIVERSITY OF CALCUTTA

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Journal of Pure Mathematics
Department of Pure Mathematics
University of Calcutta
35, Ballygunge Circular Road
Kolkata 700 019, India

Jour. Pure Math., Vol. 19, 2002, pp. 1-11

ON A GENERALIZATION OF HILBERT'S INEQUALITY

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ABSTRACT : In this paper, by introducing a parameter λ, we give a generalization of Hilbert's inequality, we also consider its equivalent form and the corresponding integral form.

2000 Mathematics Subject Classification : 26D15

Key words and phrases : Hilbert's inequality ; weight function coefficient ; Cauchy's ineqality

1. INTRODUCTION

If $0<\sum_{n=1}^{\infty} a_n^2<\infty$, and $0<\sum_{n=1}^{\infty} b_n^2<\infty$, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n}<\pi\left\{\sum_{n=1}^{\infty}a_n^2\sum_{n=1}^{\infty}b_n^2\right\}^{1/2}, \qquad \dots (1.1)$$

where the constant π is best possible (see Hardy et. al. [1]). (1.1) is well known as Hilbert's inequality, which is important in analysis and applications (see Mitrinovic [2]). Its equivalent form is

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{a_m}{m+n}\right)^2<\pi^2\sum_{n=1}^{\infty}a_n^2, \qquad \dots (1.2)$$

where the constant π^2 is best possible. The corresponding integral form of (1.1) and (1.2) are :

If $0<\int_0^{\infty} f^2(t)dt<\infty$, and $0<\int_0^{\infty} g^2(t)dt<\infty$, then

$$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{x+y}dxdy<\pi\left\{\int_0^{\infty}f^2(x)dx\int_0^{\infty}g^2(x)dx\right\}^{1/2}; \qquad \dots (1.3)$$

$$\int_0^\infty \left(\int_0^\infty \frac{f(x)}{x+y} dx \right)^2 dy < \pi^2 \int_0^\infty f^2(x) dx, \qquad \text{... (1.4)}$$

where the constant π and π^2 are still best possible.

In recent years, Gao [3,4] and Pachpatte [5] made some improvements of (1.1). By introducing a paramater λ, Yang [6] gave a generalization of (1.3) as

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda} dxdy$$

$$< B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \left\{ \int_0^\infty x^{1-\lambda} f^2(x) dx \int_0^\infty x^{1-\lambda} g^2(x) dx \right\}^{1/2}, \ (0<\lambda<1). \qquad \text{... (1.5)}$$

where the constant $B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)$ is β-function $B(p,q)$ for $p = q = \lambda/2$.

Recently, Kuang [7] proved the following inequality

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x^\lambda + y^\lambda} dxdy$$

$$< \frac{\pi}{\lambda \sin(\pi/2\lambda)} \left\{ \int_0^\infty x^{1-\lambda} f^2(x) dx \int_0^\infty x^{1-\lambda} g^2(x) dx \right\}^{1/2}, \ \left(\frac{1}{2} < \lambda < 1\right) \qquad \text{... (1.6)}$$

for $\lambda = 1$, both (1.5) and (1.6) reduce to (1.3).

In this paper we use methods similar to those of Gao and Yang [4] and Kuang [8] to generalize (1.1) with a best possible constant factor. We also consider its equivalent form which is more general than (1.6).

2. THE EXTENDED SERIES FORM

Lemma 2.1. *For $\lambda > 0$, define the weight function $\omega_\lambda(y)$ and the weight coefficient as $\tilde{\omega}_\lambda(n)$ respectively by*

$$\omega_\lambda = \int_0^\infty (x^\lambda + y^\lambda)^{-1} \left(\frac{y^\lambda}{x^\lambda}\right)^{\frac{1}{\lambda} - \frac{1}{2}} dx, \ y \in (0, \infty), \qquad \text{... (2.1)}$$

$$\tilde{\omega}_\lambda = \sum_{m=1}^{\infty} \frac{1}{m^\lambda + n^\lambda} \left(\frac{n^\lambda}{m^\lambda} \right)^{\frac{1}{\lambda} - \frac{1}{2}} n \in N. \qquad \dots (2.2)$$

Then we have $\omega_\lambda(y) = y^{1-\lambda}\pi/\lambda$, $y \in (0, \infty)$, *and for* $0 < \lambda \le 2$,

$$\tilde{\omega}_\lambda(n) < \omega_\lambda(n), \ n \in N.$$

Proof. Substituting $t = x^\lambda / y^\lambda$, we find

$$\omega_\lambda(y) = \frac{1}{\lambda}\int_0^\infty \frac{1}{y^\lambda(1+t)} \left(\frac{1}{t}\right)^{\frac{1}{\lambda}-\frac{1}{2}} y t^{\frac{1}{\lambda}-1} dt$$

$$= \frac{1}{\lambda} y^{1-\lambda} \int_0^\infty \frac{1}{(1+t)} \left(\frac{1}{t}\right)^{\frac{1}{2}} dt = \frac{\pi}{\lambda} y^{1-\lambda}.$$

Since, for $0 < \lambda \le 2$, $\frac{1}{\lambda} - \frac{1}{2} \ge 0$, then $f(x) = \frac{1}{x^\lambda + n^\lambda}\left(\frac{n^\lambda}{x^\lambda}\right)^{\frac{1}{\lambda}-\frac{1}{2}}$ is strictly decreasing ı $(0,\infty)$. Hence, we have $\tilde{\omega}_\lambda(n) < \omega_\lambda(n)$, $n \in N$. The lemma is proved. ■

Lemma 2.2. *For* $\lambda > 0$ *and* $0 < \varepsilon < \lambda/2$, *we have*

$$\int_1^\infty \frac{1}{x^{1+\varepsilon}} \int_0^{1/x^\lambda} \frac{1}{1+u} \left(\frac{1}{u}\right)^{\frac{1}{2}+\frac{\varepsilon}{2\lambda}} du dx = O(1)\left(\varepsilon \to 0^+\right). \qquad \dots (2.3)$$

Proof. Since $0 < \varepsilon < \lambda/2$, then we have

$$0 < \int_1^\infty \frac{1}{x^{1+\varepsilon}} \int_0^{1/x^\lambda} \frac{1}{1+u} \left(\frac{1}{u}\right)^{\frac{1}{2}+\frac{\varepsilon}{2\lambda}} du dx$$

$$< \int_1^\infty \frac{1}{x^{1+\varepsilon}} \int_0^{1/x^\lambda} \left(\frac{1}{u}\right)^{\frac{1}{2}+\frac{\varepsilon}{2\lambda}} du dx$$

$$< \int_1^\infty \frac{1}{x} \int_0^{1/x^\lambda} \left(\frac{1}{u}\right)^{\frac{3}{4}} du dx$$

$$= 4\int_1^\infty x^{-1-\lambda/4} dx = 16/\lambda.$$

This shows that (2.3) is valid. The lemma is proved. ■

Theorem 2.3. *If* $0 < \lambda \le 2$, $0<\sum_{n=1}^{\infty} n^{1-\lambda}a_n^2 < \infty$ *and* $0<\sum_{n=1}^{\infty} n^{1-\lambda}b_n^2 < \infty$ *then*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m^{\lambda}+n^{\lambda}} < \frac{\pi}{\lambda}\left\{\sum_{n=1}^{\infty} n^{1-\lambda}a_n^2 \sum_{n=1}^{\infty} n^{1-\lambda}b_n^2\right\}^{1/2}, \qquad \dots (2.4)$$

where the constant (π/λ) *is best possible. In particular, for* $\lambda = 1/2, 2$, *we have*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{\sqrt{m}+\sqrt{n}} < 2\pi\left\{\sum_{n=1}^{\infty}\sqrt{n}a_n^2 \sum_{n=1}^{\infty}\sqrt{n}b_n^2\right\}^{1/2}; \qquad \dots (2.5)$$

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m^2+n^2} < \frac{\pi}{2}\left\{\sum_{n=1}^{\infty}\frac{1}{n}a_n^2 \sum_{n=1}^{\infty}\frac{1}{n}b_n^2\right\}^{1/2}, \qquad \dots (2.6)$$

where the constants 2π *and* $\pi/2$ *are best possible.*

Proof. By Cauchy's inequlity and Lemma 2.1, we obtain

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_m}{m^{\lambda}+n^{\lambda}}$$

$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m}{(m^{\lambda}+n^{\lambda})^{1/2}}\left(\frac{m^{\lambda}}{n^{\lambda}}\right)^{\frac{1}{2\lambda}-\frac{1}{4}}\frac{b_n}{(m^{\lambda}+n^{\lambda})^{1/2}}\left(\frac{m^{\lambda}}{n^{\lambda}}\right)^{\frac{1}{2\lambda}-\frac{1}{4}}$$

$$\le \left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m^2}{(m^{\lambda}+n^{\lambda}}\left(\frac{m^{\lambda}}{n^{\lambda}}\right)^{\frac{1}{2\lambda}-\frac{1}{4}}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{b_m^2}{(m^{\lambda}+n^{\lambda}}\left(\frac{n^{\lambda}}{m^{\lambda}}\right)^{\frac{1}{2\lambda}-\frac{1}{4}}\right\}^{1/2}$$

$$= \left\{\sum_{m=1}^{\infty}\left[\sum_{n=1}^{\infty}\frac{1}{(m^{\lambda}+n^{\lambda})}\left(\frac{m^{\lambda}}{n^{\lambda}}\right)^{\frac{1}{2\lambda}-\frac{1}{4}}\right]a_m^2\right.$$

$$\left.\times\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\frac{1}{(m^{\lambda}+n^{\lambda})}\left(\frac{n^{\lambda}}{m^{\lambda}}\right)^{\frac{1}{2\lambda}-\frac{1}{4}}\right]b_n^2\right\}^{1/2}$$

$$= \left\{ \sum_{m=1}^{\infty} \tilde{\omega}_{\lambda}(m) a_m^2 \sum_{n=1}^{\infty} \tilde{\omega}_{\lambda}(n) b_n^2 \right\}^{1/2}$$

$$< \left\{ \sum_{m=1}^{\infty} \omega_{\lambda}(n) a_n^2 \sum_{n=1}^{\infty} \omega_{\lambda}(n) b_n^2 \right\}^{1/2}$$

$$= \frac{\pi}{\lambda} \left\{ \sum_{n=1}^{\infty} n^{1-\lambda} a_n^2 \sum_{n=1}^{\infty} n^{1-\lambda} b_n^2 \right\}^{1/2}$$

Inequality (2.4) is thus proved.

For ε ∈ (0, λ/2), setting $\tilde{a}_n = \left(\frac{1}{n^{\lambda}}\right)^{\frac{1}{\lambda} - \frac{1}{2} + \frac{\varepsilon}{2\lambda}}$ (n N), then we have

$$\sum_{n=1}^{\infty} n^{1-\lambda} \tilde{a}_n^2 = \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} \text{ and}$$

$$\frac{1}{\varepsilon} = \int_1^{\infty} \frac{1}{x^{1+\varepsilon}}\, dx < \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} = 1 + \sum_{n=2}^{\infty} \frac{1}{n^{1+\varepsilon}}$$

$$< 1 + \int_1^{\infty} \frac{1}{x^{1+\varepsilon}}\, dx = 1 + \frac{1}{\varepsilon}.$$

Hence, we find

$$\sum_{n=1}^{\infty} n^{1-\lambda} \tilde{a}_n^2 = \frac{1}{\varepsilon} + O(1) = \frac{1}{\varepsilon}(1 + o(1)) \quad (\varepsilon \to 0^+); \qquad \ldots (2.7)$$

In view of (2.3), and $\int_0^{\infty} \frac{1}{1+u}\left(\frac{1}{u}\right)^{\frac{1}{2}+\frac{\varepsilon}{2}} du = \pi + o(1) \quad (\varepsilon \to 0^+)$, we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\tilde{a}_n \tilde{b}_n}{m^{\lambda} + n^{\lambda}} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{m^{\lambda} + n^{\lambda}} \left(\frac{1}{mn}\right)^{1-\frac{\lambda}{2}+\frac{\varepsilon}{2}}$$

$$> \int_1^{\infty} \left[\int_1^{\infty} \frac{1}{x^{\lambda} + y^{\lambda}} \left(\frac{1}{xy}\right)^{1-\frac{\lambda}{2}+\frac{\varepsilon}{2}} dy \right] dx$$

(Putting $u = y^{\lambda}/x^{\lambda}$)

$$= \frac{1}{\lambda}\int_1^\infty \frac{1}{x^{1+\varepsilon}}\int_{1/x^\lambda}^\infty \frac{1}{1+u}\left(\frac{1}{u}\right)^{\frac{1}{2}+\frac{\varepsilon}{2\lambda}} dudx$$

$$= \frac{1}{\lambda}\left[\int_1^\infty \frac{1}{x^{1+\varepsilon}}\int_0^\infty \frac{1}{1+u}\left(\frac{1}{u}\right)^{\frac{1}{2}+\frac{\varepsilon}{2\lambda}} dudx\right.$$

$$\left. - \left[\int_1^\infty \frac{1}{x^{1+\varepsilon}}\int_0^{1/x^\lambda} \frac{1}{1+u}\left(\frac{1}{u}\right)^{\frac{1}{2}+\frac{\varepsilon}{2\lambda}} dudx\right]\right]$$

$$= \frac{1}{\lambda}\left[\frac{1}{\varepsilon}\int_0^\infty \frac{1}{1+u}\left(\frac{1}{u}\right)^{\frac{1}{2}+\frac{\varepsilon}{2\lambda}} du - O(1)\right]$$

$$= \frac{1}{\lambda}\left[\frac{1}{\varepsilon}(\pi + o(1) - O(1))\right] = \frac{\pi}{\lambda\varepsilon}(1+o(1)) \quad (\varepsilon \to O^+) \qquad \dots (2.8)$$

Suppose there exists a positive integer K < π/λ, such that (2.6) is valid by changing π/λ to *K*. Hence by (2.7) and (2.8), we get

$$\frac{\pi}{\lambda\varepsilon}(1+o(1)) < \sum_{m=1}^\infty\sum_{n=1}^\infty \frac{\tilde{a}_m\tilde{a}_n}{m^\lambda + n^\lambda} < K\sum_{n=1}^\infty n^{1-\lambda}\tilde{a}_n^2 = \frac{1}{\varepsilon}K(1+o(1))\ (\varepsilon \to 0^+).$$

It follows that $\pi/\lambda \le K$, which contradicts the fact that K < π/λ. Hence the constant π/λ in (2.6) is best possibel. The theorem is proved. ∎

Theorem 2.4. *If* $0 < \lambda \le 2$, $0 < \sum_{n=1}^\infty n^{1-\lambda}a_n^2 < \infty$, *then*

$$\sum_{n=1}^\infty n^{\lambda-1}\left(\sum_{m=1}^\infty \frac{a_m}{m^\lambda + n^\lambda}\right)^2 < \left(\frac{\pi}{\lambda}\right)^2 \sum_{n=1}^\infty n^{1-\lambda}a_n^2. \qquad \dots (2.9)$$

Inequality (2.9) is equivalent to (2.6). The constant $\left(\pi/\lambda\right)^2$ *in (2.9) is best possible. In particular, for* $\lambda = 1/2, 2$, *we have*

$$\sum_{n=1}^\infty \frac{1}{\sqrt{n}}\left(\sum_{m=1}^\infty \frac{a_m}{\sqrt{m}+\sqrt{n}}\right)^2 < 4\pi^2\sum_{n=1}^\infty \sqrt{n}\, a_n^2\,; \qquad \dots (2.10)$$

$$\sum_{n=1}^{\infty} n\left(\sum_{m=1}^{\infty}\frac{a_m}{m^2+n^2}\right)^2 < \frac{\pi^2}{4}\sum_{n=1}^{\infty}\frac{1}{n}a_n^2, \qquad \dots (2.11)$$

where the constants $4\pi^2$ and $\pi^2/4$ are best possible.

Proof. Since $\sum_{n=1}^{\infty} n^{1-\lambda}a_n^2 > 0$, there exists $k_0 \in N$, such that for $k > k_0$, $\sum_{m=1}^{k}\frac{|a_m|}{m^\lambda+n^\lambda} > 0$.

Putting $b_n(k) = n^{1-\lambda}\sum_{m=1}^{k}\frac{|a_m|}{m^\lambda+n^\lambda}$ $(k > k_0)$, then, by (2.6), we have

$$\begin{aligned} 0 < \sum_{n=1}^{k} n^{1-\lambda}b_n^2(k) &= \sum_{n=1}^{k} n^{\lambda-1}\left(\frac{|a_m|}{m^\lambda+n^\lambda}\right)^2 \\ &= \sum_{n=1}^{k}\sum_{m=1}^{k}\frac{|a_m|b_n(k)}{m^\lambda+n^\lambda} \\ &< \frac{\pi}{\lambda}\left\{\sum_{n=1}^{k} n^{1-\lambda}a_n^2\sum_{n=1}^{k} n^{1-\lambda}b_n^2(k)\right\}^{1/2}, \end{aligned}$$

and then

$$\sum_{n=1}^{k} n^{1-\lambda}b_n^2(k) = \sum_{n=1}^{k} n^{\lambda-1}\left(\sum_{m=1}^{k}\frac{|a_m|}{m^\lambda+n^\lambda}\right)^2 < \left(\frac{\pi}{\lambda}\right)^2\sum_{n=1}^{k} n^{1-\lambda}a_n^2. \qquad \dots (2.12)$$

Hence we find $0 < \sum_{n=1}^{\infty} n^{1-\lambda}b_n^2(\infty)$, and for , $k \to \infty$, by (2.6), we still have the inequality (2.12), and

$$\sum_{n=1}^{\infty} n^{\lambda-1}\left(\sum_{m=1}^{\infty}\frac{a_m}{m^\lambda+n^\lambda}\right)^2 \le \sum_{n=1}^{\infty} n^{\lambda-1}\left(\sum_{m=1}^{\infty}\frac{|a_m|}{m^\lambda+n^\lambda}\right)^2 < \left(\frac{\pi}{\lambda}\right)^2\sum_{n=1}^{\infty} n^{1-\lambda}a_n^2.$$

Inequality (2.9) is valid.

On the other hand, if (2.9) is valid, by Cauchy's inequality, we obtain

$$\left(\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m^\lambda+n^\lambda}\right)^2 = \left\{\sum_{n=1}^{\infty}\left[n^{(\lambda-1)/2}\sum_{m=1}^{\infty}\frac{a_m}{m^\lambda+n^\lambda}\right]\left[n^{(1-\lambda)/2}b_n\right]\right\}^2$$

$$\leq \sum_{n=1}^{\infty} n^{\lambda-1}\left(\sum_{m=1}^{\infty}\frac{a_m}{m^\lambda+n^\lambda}\right)^2 \sum_{n=1}^{\infty} n^{1-\lambda} b_n^2. \qquad \ldots (2.13)$$

By (2.9), we have (2.6). Inequality (2.9) is equivalent to (2.6).

If the constant $\left(\pi/\lambda\right)^2$ in (2.9) is not best possible, using (2.13), we may get the same result that the constant π/λ in (2.6) is not best possible, which is a contradiction. This proves the theorem. ∎

3. THE EXTENDED INTEGRAL FORM

Theorem 3.1. *If* $\lambda > 0$, $0 < \int_0^\infty t^{1-\lambda} f^2(t)\, dt < \infty$,

and $0 < \int_0^\infty t^{1-\lambda} g^2(t)\, dt < \infty$ *then*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x^\lambda + y^\lambda}\, dxdy$$

$$< \frac{\pi}{\lambda}\left\{\int_0^\infty x^{1-\lambda} f^2(x)\, dx \int_0^\infty x^{1-\lambda} g^2(x)\, dx\right\}^{1/2}; \qquad \ldots (3.1)$$

$$\int_0^\infty y^{1-\lambda}\left(\int_0^\infty \frac{f(x)}{x^\lambda + y^\lambda} dx\right)^2 dy$$

$$< \left(\frac{\pi}{\lambda}\right)^2 \int_0^\infty x^{1-\lambda} f^2(x)\, dx, \qquad \ldots (3.2)$$

where the constants π/λ *and* $\left(\pi/\lambda\right)^2$ *are best possible. Inequalities (3.1) and (3.2) are equivalent.*

Proof. By using Cauchy's ineqality, we obtain

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x^\lambda + y^\lambda} dxdy$$

$$= \int_0^\infty\int_0^\infty\left[\frac{f(x)}{(x^\lambda+y^\lambda)^{1/2}}\left(\frac{x^\lambda}{y^\lambda}\right)^{\frac{1}{2\lambda}-\frac{1}{4}}\right]\left[\frac{f(x)}{(x^\lambda+y^\lambda)^{1/2}}\left(\frac{y^\lambda}{x^\lambda}\right)^{\frac{1}{2\lambda}-\frac{1}{4}}\right]dxdy$$

$$< \left\{\int_0^\infty\int_0^\infty\left[\frac{f^2(x)}{(x^\lambda+y^\lambda)^{1/2}}\left(\frac{x^\lambda}{y^\lambda}\right)^{\frac{1}{2\lambda}-\frac{1}{4}}\right]dydx\right.$$

$$\left.\times\int_0^\infty\int_0^\infty\left[\frac{g^2(x)}{(x^\lambda+y^\lambda)^{1/2}}\left(\frac{y^\lambda}{x^\lambda}\right)^{\frac{1}{2\lambda}-\frac{1}{4}}\right]dxdy\right\}^{1/2}. \qquad \ldots (3.3)$$

If (3.3) becomes eqality, then there exists numbers A and B, such that (see Kuang [8], p. 29)

$$A\frac{f^2(x)}{(x^\lambda+y^\lambda)^{1/2}}\left(\frac{x^\lambda}{y^\lambda}\right)^{\frac{1}{2\lambda}-\frac{1}{4}} = B\frac{g^2(x)}{(x^\lambda+y^\lambda)^{1/2}}\left(\frac{y^\lambda}{x^\lambda}\right)^{\frac{1}{2\lambda}-\frac{1}{4}} \text{ a.e. in } (0,\infty)\times(0,\infty).$$

It follows that $x^{2-\lambda}f^2(x)=y^{2-\lambda}g^2(y)$ = const. a.e. in $(0, \infty) \times (0, \infty)$, which contradicts the fact that $0<\int_0^\infty t^{1-\lambda}f^2(t)\,dt<\infty$. Hence (3.3) becomes a strict ineqality, and then

$$\int_0^\infty\int_0^\infty\frac{f(x)g(y)}{x^\lambda+y^\lambda}dxdy<\left\{\int_0^\infty\omega_\lambda(x)f^2(x)\,dx\int_0^\infty\omega_\lambda(y)g^2(y)\,dy\right\}^{1/2}.$$

By Lemma 2.1, we obtain (3.1).

If the constant π/λ in (3.1) is not best possible, then there exists a positive number $K < \pi/\lambda$, such that (3.1) is valid by changing (π/λ) to K. For $\varepsilon\in(\pi/\lambda)$, setting $\tilde{f}(t)$ as:

$\tilde{f}(t) = 0$, for $t \in (0,1)$; $\tilde{f}(t) = t^{-1+\frac{\lambda}{2}-\frac{\varepsilon}{2}}$, for $t \in [1, \infty)$, by (2.8), we have

$$\frac{\pi}{\lambda\varepsilon}(1+o(1))=\int_0^\infty\int_0^\infty\frac{\tilde{f}(x)\tilde{f}(y)}{x^\lambda+y^\lambda}dxdy$$

$$< k\int_0^\infty x^{1-\lambda}\tilde{f}^2(x)dx = \frac{k}{\varepsilon}(\varepsilon \to 0^+).$$

It follows that $\pi/\lambda \le k$, which contradicts the fact that $k < \pi/\lambda$. Hence the constant π/λ in (3.1) is best possible.

There exists $T_0 > 0$, such that, for $T > T_0$, $\int_0^T \frac{|f(x)|}{(x^\lambda + y^\lambda)}dx > 0$. Writing

$$g(y,T) = y^{\lambda-1}\int_0^T \frac{|f(x)|}{(x^\lambda + y^\lambda)}dx,\ y \in (0,\infty),$$

we use (3.1) to obtain

$$\begin{aligned}
\int_0^T y^{1-\lambda}g^2(y,T)\,dy &= \int_0^T y^{\lambda-1}\left(\frac{|f(x)|}{x^\lambda + y^\lambda}dx\right)dy \\
&= \int_0^T\int_0^T \frac{|f(x)|g(y,T)}{x^\lambda + y^\lambda}\,dxdy \\
&< \frac{\pi}{\lambda}\left\{\int_0^T x^{1-\lambda}f^2(x)\,dx\int_0^T y^{1-\lambda}g^2(y,T)\,dy\right\}^{1/2} \quad \text{and}
\end{aligned}$$

$$\begin{aligned}
\int_0^T y^{1-\lambda}g^2(y,T)\,dy &= \int_0^T y^{1-\lambda}\left(\int_0^T \frac{|f(x)|}{x^\lambda + y^\lambda}\,dx\right)^2 dy \\
&< \left(\frac{\pi}{\lambda}\right)^2\int_0^T x^{1-\lambda}f^2(x)\,dx. \qquad \ldots (3.4)
\end{aligned}$$

Hence, we have $0 < \int_0^\infty y^{1-\lambda}g^2(y,\infty)\,dy < \infty$. For $T \to \infty$, still by (3.1), we have (3.4), and

$$\int_0^\infty y^{\lambda-1}\left(\int_0^\infty \frac{f(x)}{x^\lambda + y^\lambda}dx\right)^2 dy \le \int_0^\infty y^{\lambda-1}\left(\int_0^\infty \frac{|f(x)|}{x^\lambda + y^\lambda}dx\right)^2 dy < \left(\frac{\pi}{\lambda}\right)^2\int_0^\infty x^{1-\lambda}f^2(x)dx.$$

Inequality (3.2) is valid. Following the result (2.12) in Teorem 2.3, we may show that (3.1) and (3.2) are equivalent. From the equivalence of (3.1) and (3.2), it follows that the constant $\left(\pi/\lambda\right)^2$ in (3.2) is best possible. The theorem is proved. ∎

Remarks 3.2. *For $\lambda = 1$ (2.4) reduces to (1.1), so (2.4) is a new generalization of (1.1). Similarly, (2.5), (3.1) and (3.2) are new generalizations of (1.2), (1.3) and (1.4) respectively. Since the constant factor π/λ, in (3.1) for $\lambda \in (0, \infty)$, is best possible. Inequality (3.1) is more general than (1.6).*

ACKNOWLEDGEMENT

This research work was partially supported by the faculty Research Council of the University of Texas-Pan American.

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Jour. Pure Math., Vol. 19, 2002, pp. 13-17

ON THE ORDER AND TYPE OF DIFFERENTIAL MONOMIALS

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ABSTRACT : In the paper we study the relation between the order (type) of a transcendental meromorphic function and that of a differential monomial generated by it.

AMS Subject Classification (2000) : 30D35

Key words and phrases : Defferential monomials, Transcendental meromorphic function, Order, Hyper Order, Type.

1. INTRODUCTION AND DEFINITIONS

Let f be a transcendental meromorphic function defined in the open complex plane C and n_0, n_1, ...n_k, be non negative integers such that $\sum_{i=0}^{k} n_i \geq 1$. We call $p[f] = af^{n_0}(f^{(1)})^{n_1}......(f^{(k)})^{n_k}$, where T(r,a) = S(r,f) to be a differential monomial generated by f. The numbers $\gamma_p = \sum_{i=0}^{k} n_i$ and $\Gamma_p = \sum_{i=0}^{k}(i+1)n_i$ are respectively called the degree and weight of P[f].

In the paper we establish the relation between the order (type) of P[f] and f.

The following definitions are well known.

Definition 1.1. [4] For a $\in$ C $\cup$ {∞} we denote by n(r,a; f|=1) the number of simple zeros of f-a in |z|$\leq$ r. N(r,a; f|=1) is defined in terms of n(r,a; f|=1) in the usual way. Also we put

$$\delta_1(a;f) = 1 - \lim\sup_{r\to\infty} \frac{N(r, a; f|=1)}{T(r, f)}$$

Yang [3] proved that there exists at most a denumerable number of complex numbers a $\in$ C $\cup$ {∞} for which δ_1(a; f) > 0 and $\sum_{a\in C\cup\{\infty\}} \delta_1(a; f) \leq 4$

Definition 1.2. The order ρ_f and lower order λ_f of a meromorphic function f is defined as

$$\rho_f = \lim_{r\to\infty} \sup \frac{\log T(r, f)}{\log r}$$

and $$\lambda_f = \lim_{r\to\infty} \inf \frac{\log T(r, f)}{\log r}$$

if f is entire then

$$\rho_f = \lim_{r\to\infty} \sup \frac{\log^{[2]} M(r, f)}{\log r}$$

and $$\lambda_f = \lim_{r\to\infty} \inf \frac{\log^{[2]} M(r, f)}{\log r}$$

Definition 1.3. The hyper order $\bar{\rho}_f$ and hyper lower order $\bar{\lambda}_f$ of a meromorphic funciton f is defined as

$$\bar{\rho}_f = \lim_{r\to\infty} \sup \frac{\log^{[2]} T(r,f)}{\log r} \text{ and } \bar{\lambda}_f = \lim_{r\to\infty} \inf \frac{\log^{[2]} T(r,f)}{\log r}$$

If f is entire then one can easily verify that

$$\bar{\rho}_f = \lim_{r\to\infty} \sup \frac{\log^{[3]} M(r,f)}{\log r}$$

and $$\bar{\lambda}_f = \lim_{r\to\infty} \inf \frac{\log^{[3]} M(r,f)}{\log r}$$

Definition 1.4. The type of σ_f a meromorphic function f is defined as

$$\sigma_f = \lim_{r\to\infty} \sup \frac{T(r,f)}{r^{\rho_f}}, \quad 0 < \rho_f < \infty.$$

In the paper we do not explain the standard notations of value distribution theory as those are available in [1].

2. LEMMA

In this section we present a lemma which will be needed in the sequel.

Lemma 2.1. [2] Let f be of finite order or of non-zero lower order. If $\sum_{a\in C\cup\{\infty\}} \delta_1(a, f) = 4$

then $\lim_{r\to\infty} \frac{T(r,P[f])}{T(r,f)} = \Gamma_p - (\Gamma_p - \gamma_p)\,\theta(\infty, f)$.

3. THEOREMS

In this section we present the main results of the paper.

Theorem 3.1. Let f be of positive finite order and $\sum_{a\in C\cup\{\infty\}} \delta_1(a; f) = 4$ Then the order of P[f] is same as that of f and type of P[f] is $\{\Gamma_p - (\Gamma_p - \gamma_p)\,\theta(\infty, f)\}$ times that of f.

Proof. Let ρ_1, ρ_2 be the orders and τ_1, τ_2 be the types of f and P[f] respectively. Then by Lemma 2.1 we get

$$\rho_1 = \limsup_{r\to\infty} \frac{\log T(r,P[f])}{\log r}$$

$$= \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r} . \frac{\log T(r,P[f])}{\log T(r,f)}$$

$$\leq \rho_1 \limsup_{r\to\infty} \frac{\log T(r,P[f])}{\log T(r,f)}$$

$$= \rho_1 \limsup_{r\to\infty} \frac{\log \frac{T(r,P[f])}{T(r,f)} + \log T(r,f)}{\log T(r,f)}$$

$$= \rho_1 \qquad \dots (1)$$

Again $\rho_1 = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}$

$$\leq \rho_2 \limsup_{r\to\infty} \frac{\log \frac{T(r,f)}{T(r,P[f])} + \log T(r,P[f])}{\log T(r,[f])}$$

$$= \rho_2 \qquad \dots (2)$$

From (1) and (2) we get $\rho_1 = \rho_2$.

Now by Lemma 2.1 we see that

$$\tau_2 = \limsup_{r\to\infty} \frac{T(r,P[f])}{r^{\rho_2}}$$

$$= \limsup_{r\to\infty} \frac{T(r,f)}{r^{\rho_1}} \cdot \lim_{r\to\infty} \frac{T(r,P[f])}{T(r,f)}$$

$$= \{\Gamma_p - (\Gamma_p - \gamma_p)\, \theta(\infty, f)\}\tau_1.$$

This proves the theorem.

Theorem 3.2. Let f be of finite order or of non zero lower order. If $\sum_{a\in C\cup\{\infty\}} \delta_1(a;\, f) = 4$ then the lower order of P [f] is same as that of f.

The proof is omitted.

Theorem 3.3. Let f be of finite order or of non zero lower order. Also let $\sum_{a\in C\cup\{\infty\}} \delta_1(a;\, f) = 4$ Then the hyper order of P [f] is same as that of f.

Proof. Let $\overline{\rho}_1$, $\overline{\rho}_2$ be the hyper orders of f and P [f] respectively.

Now in view of Lemma 2.1, $\lim_{r\to\infty} \frac{\log^{[2]} T(r,\, P[f])}{\log^{[2]} T(r,f)}$ exists and is equal to 1.

Thus we get,

$$\overline{\rho}_2 = \limsup_{r\to\infty} \frac{\log^{[2]} T(r,\, P[f])}{\log r}$$

$$= \limsup_{r\to\infty} \left\{ \frac{\log^2 T(r,f)}{\log r} \cdot \frac{\log^{[2]} T(r,\, P[f])}{\log^2 T(r,f)} \right\}$$

$$= \limsup_{r\to\infty} \frac{\log^2 T(r,f)}{\log r} \cdot \lim_{r\to\infty} \frac{\log^{[2]} T(r,\, P[f])}{\log^{[2]} T(r,f)}$$

$$= \overline{\rho}_1.$$

Theorem 3.4. Let f be of finite order or of non zero lower order and $\sum_{a \in C \cup \{\infty\}} \delta_1(a; f) = 4$. Then the hyper lower orders of P[f] and f are same.

The proof is omitted.

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Jour. Pure Math., Vol. 19, 2002, pp. 19-29

FURTHER GROWTH PROPERTIES OF COMPOSITE ENTIRE AND MEROMORPHIC FUNCTIONS

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ABSTRACT : In the paper we study the growth of composite entire and meromorphic functions which improve some known results.

AMS Subject Classification (2000) : 30D35, 30D30

Key words and phrases : Entire function, meromorphic function, composition, growth, order.

1. INTRODUCTION AND DEFINITIONS

Let f and g be two transcendental entire functions defined in the open complex plane C. It is well known [2] that $\lim_{r\to\infty}\frac{T(r, fog)}{T(r,f)}=\infty$ and $\lim_{r\to\infty}\frac{T(r, fog)}{T(r,g)}=\infty$. Singh [8] proved some comparative growth properties of log T(r, fog) and T(r, f). He [8] also raised the question of investigating the comparative growth of log T(r, fog) and T(r, g) which he was unable to solve. However, some results on the comparative growth of log T(r,fog) and T(r,g) are proved in [4]. In the paper we further investigate the above question of Singh [8] and study the comparative growth of log T(r, fog) with $T(r,f)\{\log T(r,f)\}^k$ and $T(r,g)\{\log T(r,g)\}^k$ where f is taken to be meromorphic, g is entire and k > 0.

If f and g are of positive lower order then Song and Yang [10] proved that

$$\lim_{r\to\infty}\frac{\log^{[2]}M(r, fog)}{\log^{[2]}M(r, f)}=\lim_{r\to\infty}\frac{\log^{[2]}M(r, fog)}{\log^{[2]}M(r, g)}=\infty, \text{ where}$$

$\log^{[k]}x=\log(\log^{[k-1]}x)$ for k = 1,2,3, ... and $\log^{[0]}x = x$.

Also in the sequel we use the following notation :

$\exp^{[k]}x = \exp(\exp^{[k-1]}x)$ for k = 1,2,3, ... and $\exp^{[0]}x = x$.

Since M(r, f) and M(r, g) are increasing function of r, Singh and Baloria [9] asked whether for sufficiently large R = R(r)

$$\limsup_{r\to\infty}\frac{\log^{[2]} M(r, fog)}{\log^{[2]} M(R, f)} < \infty \text{ and } \limsup_{r\to\infty}\frac{\log^{[2]} M(r, fog)}{\log^{[2]} M(R, g)} < \infty,$$

Singh and Baloria [9], Lahiri and Sharma [6], Liao and Yang [7] worked on this question. In the paper we throw some light on the comparative growth of $\log^{[2]}M(r, fog)$ and $\log M(r, g)$ where f and g are any two entire functions. We also study about the estimation of lower order of a composite meromorphic function whose left factor is of zero order. We do not explain the standard notations and definitions of the theory of entire and meromorphic functions as those are availabe in [11] and [3].

Definitions 1.1. The order ρ_f and lower order λ_f of a meromorphic function is defined as

$$\rho_f = \limsup_{r\to\infty}\frac{\log T(r,f)}{\log r} \text{ and } \lambda_f = \liminf_{r\to\infty}\frac{\log T(r,f)}{\log r}$$

If f is entire, one can easily verify that

$$\rho_f = \limsup_{r\to\infty}\frac{\log^{[2]} M(r,f)}{\log r} \text{ and } \lambda_f = \liminf_{r\to\infty}\frac{\log^{[2]} M(r,f)}{\log r}.$$

Definition 1.2. [7] Let f be a meromorphic function of order zero. Then ρ_f and λ_f are defined as follows :

$$\rho^*_f = \limsup_{r\to\infty}\frac{\log T(r,f)}{\log^{[2]} r} \text{ and } \lambda^*_f = \liminf_{r\to\infty}\frac{\log T(r,f)}{\log^{[2]} r}.$$

If f is entire, then clearly

$$\rho^*_f = \limsup_{r\to\infty}\frac{\log^{[3]} M(r,f)}{\log^{[2]} r} \text{ and } \lambda^*_f = \liminf_{r\to\infty}\frac{\log^{[3]} M(r,f)}{\log^{[2]} r}$$

Definition 1.3. The type σ_f of an entire function f is defined as

$$\rho_f = \limsup_{r\to\infty}\frac{\log M(r,f)}{r^{\rho_f}},\ 0 < \rho_f < \infty.$$

2. LEMMAS

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.1. [2] If f and g are entire functions, for all sufficienctly large values or r,

$$M(r, fog) \geq M(\tfrac{1}{8}M(\tfrac{r}{2}, g) - |g(0)|, f).$$

Lemma 2.2. If f is meromorphic and g is entire then for all sufficiently large values of r,

Lemma 2.3. [5]. If f is a non constant entire function of finite order then

$$\liminf_{r\to\infty} \frac{\log M(r,f)}{T(r, f)\{\log T(r, f)\}^k} = 0 \text{ where } k > 0.$$

Lemma 2.4. Let f be an entire function such that $0 < \rho_f < \infty$. If σ_f and $\sigma_f^{(k)}$ be the respective types of f and $f^{(k)}$ then $\sigma_f^{(k)} \le (2^k)^{\rho_f} \sigma_f$ where $K = 0, 1, 2, 3, \ldots\ldots$.

Proof. It is known from G. Valiron {[11], p.35} that

$$\frac{1}{r}\{M(r,f) - |f(0)|\} \le M(r,f) \le \frac{1}{r} M(2r, f).$$

Noting that $\rho_f^{(k)} = \rho_f$ we get from the second part of the inequality for $r \ge 1$,

$$M(r,f^{(k)}) \le M(2^k r, f)$$

$$\text{or,} \quad \frac{\log M(r,f^{(k)})}{r^{\rho_{f^{(k)}}}} \le \frac{\log M(2^k r, f)}{(2^k r)^{\rho_f}} . (2^k)^{\rho_f}$$

$$\text{or,} \quad \limsup_{r\to\infty} \frac{\log M(r,f^{(k)})}{r^{\rho_{f^{(k)}}}} \le (2^k)^{\rho_f} \limsup_{r\to\infty} \frac{\log M(2^k r, f)}{(2^k r)^{\rho_f}}$$

or, $\sigma_f^{(k)} \le (2^k)^{\rho_f} \sigma_f$, which proves the lemma.

Lemma 2.5. Let f be meromorphic and g be entire such that $\lambda_g < \infty$. If $\lambda_{fog} = \infty$, then for every positive number A,

$$\lim_{r\to\infty} \frac{\log T(r, fog)}{\log^{[2]} M(r^A, g^{(k)})} = \infty \text{ where } k = 0, 1, 2, 3, \ldots\ldots$$

Proof. Let us assume that the conclusion of the lemma does not hold.

Then there exists a constant $B > 0$ such that

$$\lim_{r\to\infty} \frac{\log T(r, fog)}{\log^{[2]} M(r^B, g^{(k)})} = \mu < \infty,$$ provided the limit exists. Then for all larger r,

$$\log T(r, fog) < (\mu + \varepsilon) \log^{[2]} M(r^B, g^{(k)}) \quad \ldots (1)$$

Again, for a sequence of values of r, tending to infinity,

$$\log^{[2]} M(r^B, g^{(k)}) < (\lambda_g(k) + \varepsilon)\, B \log r = (\lambda_g + \varepsilon)\, B \log r \qquad \ldots (2)$$

Thus combining (1) and (2) we get for a sequence of values of r tending to infinity

$\log T(r, fog) < (\mu + \varepsilon)(\lambda_g + \varepsilon)\, B \log r$

which implies that $\lambda_{fog} < \infty$. This is a contradiction.

Thus the lemma is proved.

3. THEOREMS

In this section we present the main results of the paper.

Theorem 3.1. Let f be meromorphic and g be non constant entire such that ρ_f and ρ_g are finite. Then

$$\liminf_{r\to\infty} \frac{\log T(r, fog)}{T(r, g)\{\log T(r, g)\}^k} = 0 \text{ where } k > 0.$$

Proof. By Lemma 2.2 and $T(r, g) \le \log^+ M(r, g)$ we get for all sufficiently large values of r,

$$\log T(r,fog) \le \log T(M(r,g), f) + \log \{1 + 0(1)\}$$

$$\text{or,} \quad \log T(r,fog) < (\rho_f + \varepsilon) \log M(r,g) + \log \{1 + 0(1)\}$$

$$\text{or,} \quad \frac{\log T(r,fog)}{T(r,g)\{\log T(r,g)\}^k} < \frac{(\rho_f + \varepsilon)\log M(r,g) + \log\{1 + 0(1)\}}{T(r,g)\{\log T(r,g)\}^k} \qquad \ldots (3)$$

Now by Lemma 2.3 it follows from (3),

$$\liminf_{r\to\infty} \frac{\log T(r,fog)}{T(r,g)\{\log T(r,g)\}} = 0.$$

This proves the theorem.

Remark 3.2. Considering f = g = exp z one can easily verify that no term in the denominator of $\frac{\log T(r,fog)}{T(r,g)\{\log T(r,g)\}^k}$ can be removed.

Remark 3.3. The condition $\rho_f < \infty$ in Theorem 3.1 is necessary which is evident from the following example.

Example 3.4. Let $f = \exp^{[2]} z$, $g = z$ and $k = 1$

Then and $\rho_f = \infty$ and $\rho_g = 0$.

Since T(r, fog) ~ $\dfrac{e^r}{(2\pi^3 r)^{1/2}}$ and

T(r, g) $\leq$ log M(r, g) = log r it follows that

$$\frac{\log T(r,fog)}{T(r,g)\{\log T(r,g)\}^k} \geq \frac{r-\frac{1}{2}\log r+O(1)}{\log r\{\log^{[2]} r\}}$$

which implies that

$$\lim_{r\to\infty}\frac{\log T(r,fog)}{T(r,g)\{\log T(r,g)\}^k} = \infty.$$

Theorem 3.5. Let f and g be two entire functions such that ρ_f and ρ_g are finite.

Also let $\lambda_f > \rho_g$. Then $\liminf\limits_{r\to\infty}\dfrac{\log T(r,fog)}{T(r,f)\{\log T(r,f)\}^k} = 0$

Proof. Since $\lambda_g < \rho_f$ we can choose $\varepsilon(>0)$ in such a way that $\rho_g + \varepsilon < \lambda_f - \varepsilon$. By Lemma 2.2 and T(r, g) $\leq$ $\log^+$M(r,g) we obtain fro all sufficiently large values of r,

log T(r,fog) < (ρ_f + ε) log M(r,g) + log{1 + O(1)}

or, $$\frac{\log T(r,fog)}{T(r,f)\{\log T(r,f)\}^k}$$

$$< (\rho_f + \varepsilon)\frac{\log M(r,f)}{T(r,f)\{\log T(r,f)\}^k}\cdot\frac{\log M(r,g)}{\log M(r,f)} + \frac{\log\{1+o(1)\}}{T(r,f)\{\log T(r,f)\}^k} \quad \ldots (4)$$

Again for all sufficiently large values of r,

log M(r,g) < $r^{\rho_g + \varepsilon}$ and log M(r,f) > $r^{\rho_g - \varepsilon}$.

Thus from (4) we obtain.

$$\frac{\log T(r,fog)}{T(r,f)\{\log T(r,f)\}^k}$$

$$\leq (\rho_f + \varepsilon)\frac{\log M(r,f)}{T(r,f)\{\log T(r,f)\}^k}\cdot\frac{r^{\rho_g+\varepsilon}}{r^{\lambda_f-\varepsilon}} + \frac{\log\{1+o(1)\}}{T(r,f)\{\log T(r,f)\}^k}$$

$$\text{or,} \quad \liminf_{r\to\infty} \frac{\log T(r,fog)}{T(r,f)\{\log T(r,f)\}^k}$$

$$\leq \quad (\rho_f + \varepsilon) \liminf_{r\to\infty} \frac{\log M(r,f)}{T(r,f)\{\log T(r,f)\}^k} \cdot \lim_{r\to\infty} \frac{r^{\rho_g+\varepsilon}}{r^{\lambda_f-\varepsilon}} \quad \dots (5)$$

Now by Lemma 2.3, the theorem follows from (5).

Remarks 3.6. The following example shows that the condition $\lambda_f > \rho_g$ in Theorem 3.5 necessary.

Example 3.7. Let f = exp z, g = exp (z^2) and k = 1.

Then $\lambda_f = \rho_f = 1$ and $\rho_g = 2$.,

Now 3T (2r,fog) $\geq$ log M(r,fog) = exp(r^2)

$$\text{or,} \quad \log T(r,fog) \geq \frac{r^2}{4} + O(1)$$

$$\text{and} \quad T(r,f) = \frac{r}{\pi}.$$

Thus it follows that

$$\frac{\log T(r,fog)}{T(r,f)\{\log T(r,f)\}^k} \geq \frac{\left(\frac{r^2}{4}\right) + O(1)}{\left(\frac{r}{\pi}\right)\{\log r + O(1)\}}$$

$$\text{i.e.} \quad \lim_{r\to\infty} \frac{\log T(r,fog)}{T(r,f)\{\log T(r,f)\}^k} = \infty$$

Theorem 3.8. Let f and g be two entire functions such that $0 < \lambda_f < \infty$ and $0 < \rho_g < \infty$. Also let $0 < \sigma_g < \infty$. Then

$$\limsup_{r\to\infty} \frac{\log^{[2]} M(r,fog)}{\log M(r,g^{(k)})} \geq \frac{\lambda_f}{2^{(k+1)\rho_g}}$$

where K = 0, 1, 2, 3, ...

Proof. Let $0 < \varepsilon < \min\{\lambda_f, \delta_g\}$. Then for a sequence of values of r tending to infinity we obtain.

$$Log M\left(\tfrac{r}{2}, g\right) > (\sigma_g - \varepsilon)\left(\tfrac{r}{2}\right)^{\rho_g} \quad \dots (6)$$

Again from Lemma 2.1 we get for all sufficiently large values of r,

$$\log^{[2]} M(r, fog) > (\lambda_f - \varepsilon) \log\tfrac{1}{16} + (\lambda_f - \varepsilon) \log M\left(\tfrac{r}{2}, g\right) \quad \ldots (7)$$

Now for a sequence of values of r tending to infinity it follows from (6) and (7)

$$\log^{[2]} M(r, fog) > (\lambda_f - \varepsilon) \log\tfrac{1}{16} + (\lambda_f - \varepsilon)(\sigma_g - \varepsilon)\left(\tfrac{r}{2}\right)^{\rho_g} \quad \ldots (8)$$

Again by Lemma 2.4 we get for all large values of r,

$$\log M(r, g^{(k)}) < (\sigma_g^{(k)} + \varepsilon) r^{\rho_g} \leq (2^{k\rho_g} \sigma_g + \varepsilon) r^{\rho_g} \quad \ldots (9)$$

So from (8) and (9) it follows for a sequence of values of r tending to infinity,

$$\frac{\log^{[2]} M(r, fog)}{\log M(r, g^{(k)})} > \frac{(\lambda_f - \varepsilon)\log\frac{1}{16} + (\lambda_f - \varepsilon)(\sigma_g - \varepsilon)\left(\frac{r}{2}\right)^{\rho_g}}{(2^{k\rho_g} \sigma_g + \varepsilon) r^{\rho_g}} \quad \ldots (10)$$

Since $\varepsilon(>0)$ is arbitrary we get from (10)

$$\limsup_{r \to \infty} \frac{\log^{[2]} M(r, fog)}{\log M(r, g^{(k)})} \geq \frac{\lambda_f}{2^{(k+1)\rho_g}} \quad \ldots (13)$$

This proves the theorem.

Theorem 3.9. Let f be meromorphic and g be entire such that $0 < \lambda_{fog} \leq \rho_{fog} < \infty$ and $0 < \lambda_g \leq \rho_g < \infty$. Then for any positive number A,

$$\frac{\lambda_{fog}}{A\rho_g} \leq \liminf_{r \to \infty} \frac{\log T(r, fog)}{\log T(r^A, g^{(k)})} \leq \frac{\lambda_{fog}}{A\lambda_g}$$

$$\leq \limsup_{r \to \infty} \frac{\log T(r, fog)}{\log T(r, g^{(k)})} \leq \frac{\rho_{fog}}{A\lambda_g} \quad \text{where } k = 0, 1, 2, \ldots\ldots .$$

Proof. From the definition of order and lower order we have for arbitary positive ε and for all large values of r,

$$\log T(r, fog) > (\lambda_{fog} - \varepsilon) \log r \quad \ldots (11)$$

$$\text{and} \quad \log T(r^A, g^{(k)}) < A(\rho_g^{(k)} + \varepsilon) \log r = A(\rho_g + \varepsilon) \log r \quad \ldots (12)$$

Now from (11) and (12) it follows for all large values of r,

$$\frac{\log T(r,fog)}{\log T(r^A,g^{(k)})} > \frac{\lambda_{fog}-\varepsilon}{A(\rho_g+\varepsilon)}$$

As ε (> 0) is arbitary, we obtain

$$\liminf_{r\to\infty}\frac{\log T(r,fog)}{\log T(r^A,g^{(k)})} \geq \frac{\lambda_{fog}}{A\rho_g} \qquad \ldots (13)$$

Again for a sequence of values of r tending to infinity,

$$\log T(r,fog) < (\lambda_{fog}+\varepsilon)\log r \qquad \ldots (14)$$

and for all large values of r,

$$\log T(r^A,g^{(k)}) > A(\lambda_g-\varepsilon)\log r \qquad \ldots (15)$$

So combining (14) and (15) we get for a sequence of values of r tending to infinity,

$$\frac{\log T(r,fog)}{\log T(r^A,g^{(k)})} < \frac{\lambda_{fog}+\varepsilon}{A(\lambda_g-\varepsilon)}.$$

Since ε (> 0) is arbitrary it follows that,

$$\liminf_{r\to\infty}\frac{\log T(r,fog)}{\log T(r^A,g^{(k)})} \leq \frac{\lambda_{fog}}{A\lambda_g} \qquad \ldots (16)$$

Also for a sequence of values of r tending to infinity,

$$\log T(r^A,g^{(k)}) < A(\lambda_g+\varepsilon)\log r \qquad \ldots (17)$$

Now from (11) and (17) we obtain for a sequence of values of r tending to infinity,

$$\frac{\log T(r,fog)}{\log T(r^A,g^{(k)})} > \frac{\lambda_{fog}-\varepsilon}{A(\lambda_g+\varepsilon)}$$

Choosing $\varepsilon \to 0$ We get

$$\limsup_{r\to\infty}\frac{\log T(r,fog)}{\log T(r^A,g^{(k)})} \geq \frac{\lambda_{fog}}{A\lambda_g} \qquad \ldots (18)$$

Also for all large values of r,

$$\log T(r,fog) < (\rho_{fog}+\varepsilon)\log r \qquad \ldots (19)$$

So from (15) and (19) it follows for all large values of r,

$$\frac{\log T(r,fog)}{\log T(r^A,g^{(k)})} < \frac{\rho_{fog}+\varepsilon}{A(\lambda_g-\varepsilon)}$$

As ε (> 0) is arbitrary we obtain

$$\limsup_{r\to\infty}\frac{\log T(r,fog)}{\log T(r^A,g^{(k)})} \le \frac{\rho_{fog}}{A\lambda_g} \qquad \ldots (20)$$

Thus the theorem follows from (13), (16), (18) and (20).

Theorem 3.10. Let f be meromorphic and g be entire such that $0<\lambda_{fog}\le\rho_{fog}<\infty$ and $0<\rho_g<\infty$. Then for any positive number A,

$$\liminf_{r\to\infty}\frac{\log T(r,fog)}{\log T(r^A,g^{(k)})} \le \frac{\rho_{fog}}{A\rho_g} \le \limsup_{r\to\infty}\frac{\log T(r,fog)}{\log T(r^A,g^{(k)})}$$

where k = 0, 1, 2, ...

Proof. From the definition of order we get for a sequence of values of r tending to infinity,

$$\log T(r^A,g^{(k)}) > A(\rho_g-\varepsilon)\log r \qquad \ldots (21)$$

Now from (19) and (21) it follows for a sequence of values of r tending to infinity,

$$\frac{\log T(r,fog)}{\log T(r^A,g^{(k)})} < \frac{\rho_{fog}+\varepsilon}{A(\rho_g-\varepsilon)}$$

As ε (> 0) is arbitrary we obtain,

$$\liminf_{r\to\infty}\frac{\log T(r,fog)}{\log T(r^A,g^{(k)})} \le \frac{\rho_{fog}}{A\rho_g} \qquad \ldots (22)$$

Again, for a sequence of values of r tending to infinity,

$$\log T(r,fog) > (\rho_{fog}-\varepsilon)\log r. \qquad \ldots (23)$$

So combining (12) and (23) we get for a sequence of values of r tending to infinity,

$$\frac{\log T(r,fog)}{\log T(r^A,g^{(k)})} > \frac{\rho_{fog}-\varepsilon}{A(\rho_g+\varepsilon)}.$$

Since ε (> 0) is arbitrary it follows that,

$$\limsup_{r\to\infty} \frac{\log T(r,fog)}{\log T(r^A, g^{(k)})} \geq \frac{\rho_{fog}}{A\rho_g} \qquad \ldots (24)$$

Thus the theorem follows from (22) and (24).

In the next theorem we estimate the lower order of a composite meromorphic function whose left factor is of zero order.

Theorem 3.11. Let f be a meromorphic function of order zero and g be an entire function such that $\lambda_g < \infty$. If $\lambda_f^* < \infty$ then $\lambda_{fog} < \infty$.

Proof. By Lemma 2.2 and the inequality T(r,g) ≤ $\log^+$ M(r,g)

we obtain for all sufficiently large values of r,

$$\frac{\log T(r,fog)}{\log^{[2]} M(r,g)} \leq \frac{\log T(M(r,g),f) + O(1)}{\log^{[2]} M(r,g)}.$$

$$\text{or,} \quad \liminf_{r\to\infty} \frac{\log T(r,fog)}{\log^{[2]} M(r,g)} \leq \lambda_f^* < \infty \qquad \ldots (25)$$

by the given condition.

Now by Lemma 2.5 and (25) it follows that $\lambda_{fog} < \infty$. This proves the theorem.

ACKNOWLEDGEMENT

The author is grateful to the referee for his / her valuable suggestions towards the improvement of the paper.

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Jour. Pure Math., Vol. 19, 2002, pp. 31-39

STRONGLY θ-β-CONTINUOUS FUNCTIONS

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ABSTRACT : In the paper, we introduce a new class of functions called strongly θ-β continuous functions which is stronger than β-continuous functions and investigate their properties.

AMS Subject Classification : 54C08

Key words and phrases : β-open, semi-preopen, semi-pre θ-open, semi-pre θ-closure, β-continuous, strongly θ-β-continuous.

1. INTRODUCTION

A subset A of topological space X is called β-open [1] or semi-preopen [3] if $A \subset Cl(\text{Int}(\text{Cl}(A)))$. A function $f : X \to Y$ is called β-continuous [1] if the preimage $f^{-1}(V)$ of each open set V of Y is β-open in X. Borsik and Doboš [5] introduced the notion of almost quasicontinuous functions to obtain decompositions of quasi continuity. Borsik [4], Ewert [6], and Popa and Noiri [14] independently showed that β-continuity and almost quasicontinuity are equivalent of each other. Popa and Noiri [15] intoduced and investigated weakly β-continuous functions which are called weakly semi-precontinuous functions by Ghosh and Bhattacharyya [7]. Noiri and Popa [12] introduced and investigated almost β-continuity. The purpose of the present paper is to introduce and investigate a stronger form of β-continuity called strongly θ-β-continuous functions.

2. PRELIMINARIES

Throughout the present paper, X and Y denote topological spaces. Let S be a subset of X. We denote the interior and the closure of a set S by Int (S) and Cl(S), respectively. A subset S is said to be β-open [1] or *semi-preopen* [3] (resp. *α-open*[9]) if $S \subset \text{Cl}(\text{Int}(\text{Cl}(S)))$ (resp. $S \subset \text{Int}(\text{Cl}(\text{Int}(S)))$). The complement of a β-open (resp. semi-preopen) set is called β-closed (resp semi-preclosed). The intersection of all semi-preclosed sets containing S is called the semi-preclosure [3] of S and is denoted by spCl(S). The *semi-preinterior* of S is defined by the union of all *semi-preopen* sets contained in S and is denoted by spInt(S). The family of all semi-preopen sets of X is denoted by SPO(X). We set SPO$(X,x) = \{U : x \in U$ and $U \in$ SPO $(X)\}$. A point x of X is called a *θ-cluster point* of S if $\text{Cl}(U) \cap S \neq \phi$ for every open set U of X containing x. The set of all θ-cluster points of S is called the *θ-closure* of S and is denoted by $\text{Cl}_\theta(S)$. A subset S is said to be *semi-pre-θ-closed* [16] if $S = \text{Cl}_\theta(S)$.

The complement of a θ-closed set is said to be *θ-open.* A point x of X is called a semi-pre *θ-cluster point* of S if $\mathrm{spCl}(U) \cap S \neq$ for every semi-preopen set U of X containing x. The set of all semi-pre θ-cluster points of S is called the *semi-pre θ-closure* (briefly *sp-θ-closure* of S) and is denoted by $\mathrm{spCl}_\theta(S)$. A subset S is said to be *semi-pre-θ-closed* (briefly *sp-θ*-closed) if $S = \mathrm{spCl}_\theta(S)$. The complement of a *semi-pre-θ-closed* set is said to be *semi-pre-θ-open* (briefly *sp-θ-open*).

Definition 2.1. A function $f : X \to Y$ is said to be

(1) *β-continuous* [1] or *almost quasicontinuous* [5] if $f^{-1}(V) \in \mathrm{SPO}(X)$ for each open set V or Y,

(2) *Weakly β-continuous* [15] (resp. *almost β-continuous* [12] if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists $U \in \mathrm{SPO}(X,x)$ such that $f(U) \subset \mathrm{Cl}(V)$ (resp. $f(U) \subset \mathrm{Int}\ (\mathrm{Cl}(V))$).

Definition 2.2. A function $f : X \to Y$ is said to be *strongly θ-β-continuous* (briefly *st.θ.β*.c.) if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists $U \in \mathrm{SPO}(X,x)$ such that $f(\mathrm{spCl}(U)) \subset V$.

Definition 2.3. A function $f : X \to Y$ is said to be *strongly-θ-continuous* [10] if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists an open neighbourhood U of x such that $f(\mathrm{Cl}(U)) \subset V$.

Remarks 2.5. (1) Strong θ-β continuity is stronger than β-continuity and is weaker than strong θ-continuity.

(2) Strong θ-β-continuity and continuity are independent of each other as the following simple examples show.

Example 2.6. Let $X = \{a,b,c\}$, $\tau = \{\Phi, X, \{a,b\}\}$ and $\sigma = \{\Phi, X, \{c\}\}$. Define a function $f : (X, \tau) \to (X, \sigma)$ as follows : $f(a) = a$, $f(b) = f(c) = c$. Then f is *st.θ.β*.c. but it is not continuous.

Example 2.7. Let $X = \{a,b,c\}$ and $\tau = \{\Phi, X\{a\}, \{a,b\}\}$, Then, the identity function $f : (X, \tau) \to (X,\tau)$ is continuous but not *st.θ.β*.c. at a.

3. CHARACTERIZATIONS

Theorem 3.1. *For a function $f : X \to Y$, The following properties are equivalent :*

(1) *f is strongly θ-β-continuous ;*

(2) *$f^{-1}(V)$ is sp-θ-open in X for every open set V of Y ;*

(3) *$f^{-1}(F)$ is sp-θ-closed in X for every closed set F of Y ;*

(4) $f(\mathrm{spCl}_\theta(A)) \subset \mathrm{Cl}(f(A))$ *for every subset A of X ;*

(5) $\mathrm{spCl}_\theta(f^{-1}(B)) \subset f^{-1}\mathrm{Cl}(f(B))$ *for every subset B of Y.*

Proof. (1) ⇒ (2) : Let V be any open set of Y. Suppose that $x \in f^{-1}(V)$. There exists $U \in \mathrm{SPO}(X,x)$ such that $f(\mathrm{spCl}(U)) \subset V$. Therefore, we have $x \in U \subset \mathrm{spCl}(U) \subset f^{-1}(V)$. This shows that $f^{-1}(V)$ is sp-θ-open in X.

(2) ⇒ (3) : This is obvious.

(3) ⇒ (4) : Let A be any subset of X. Since $\mathrm{Cl}(f(A))$ is closed in Y, by (3) $f^{-1}(\mathrm{Cl}(f(A)))$ is sp-θ-closed and we have

$$\mathrm{spCl}_\theta(A) \subset \mathrm{spCl}_\theta(f^{-1}(f(A))) \subset \mathrm{spCl}_\theta(f^{-1}(\mathrm{Cl}(f(A))) = f^{-1}(\mathrm{Cl}(f(A))).$$

Therefore, we obtain $f(\mathrm{spCl}_\theta(A)) \subset \mathrm{Cl}((f(A))$.

(4) ⇒ (5) : Let B be any subset of Y. By (4), we obtain $f(\mathrm{spCl}_\theta(f^{-1}(B))))$ $\subset Cl(f(f^{-1}(B))) \subset \mathrm{Cl}(B)$ and hence $\mathrm{spCl}_\theta(f^{-1}(f(B)) \subset f^{-1}(\mathrm{Cl}(B))$.

(5) ⇒ (1) : Let $x \in X$ and V be any open neighborhood of $f(x)$. Since $Y - V$ is closed in Y, we have $\mathrm{spCl}_\theta(f^{-1}(Y - V)) \subset f^{-1}(\mathrm{Cl}(Y - V)) = f^{-1}(Y - V)$. Therefore, $f^{-1}(Y - V)$ is an *sp-θ*-closed in X and $f^{-1}(V)$ is an *sp-θ*-open set containing x. There exists $U \in \mathrm{SPO}(X,x)$ such that $\mathrm{SpCl}(U) \subset f^{-1}(V)$; hence $f(\mathrm{spCl}(U)) \subset V$. This shows that f is *st.θ.β*.c.

Definition 3.2. A function $f : X \to Y$ is said to be *faintly β-continuous* [11] if for each point $x \in X$ and each θ-open set V containin $f(x)$, there exists $U \in \mathrm{SPO}(X,x)$ such that $f(U) \subset V$.

Theorem 3.3. *Let Y be a regular space. Then, for a function $f : X \to Y$ the following properties are equivalent :*

(1) *f is faintly β-continuous ;*

(2) *f is weakly β-continuous ;*

(3) *f is almost β-continuous ;*

(4) *f is β-continuous ;*

(5) *f is st.θ.β*.c.

Proof. It is shown in [11] that (1), (2) and (4) are equivalent. Since it is obvious that (5) implies (4), we shall show that (4) implies (5).

(4) $\Rightarrow$ (5) : Let $x \in X$ and V be an open set Y containing $f(x)$. Since Y is regular, there exists an open set W such that $f(x) \in W \subset \text{Cl}(W) \subset V$. Since f is β-continuous, there exists $U \in \text{SPO}(X,x)$ such that $f(U) \subset W$. We shall show that $f(\text{spCl}(U)) \subset \text{Cl}(W)$. Suppose that $y \notin \text{Cl}(W)$. There exists an open neighborhood G of y such that $G \cap W = \Phi$. Since f is β-continuous, $f^{-1}(G) \in \text{SPO}(X)$ and $f^{-1}(G) \cap U = \Phi$ and hence $f^{-1}(G) \cap \text{SpCl}(U) = \Phi$. Therefore, we obtain $G \cap f(\text{spcl}(U)) = \Phi$ and $y \notin f(\text{spCl}(U))$. Consequently, we have $f(\text{spCl}(U) \subset \text{Cl}(W) \subset V$. This shows that f is st.$\theta.\beta$.c.

Definition 3.4. A space X is said to be *semipre-regular* (resp. *β-regular* [2] or *sp-regular* [13] if for each semi-preclosed (resp. closed) set F and each point $x \in X - F$, there exist disjoint semi-preopen sets U and V such that $x \in U$ and $F \subset V$.

Theorem 3.5. *A continuous function $f : X \to Y$ is st.$\theta.\beta$.c. if and only if X is β-regular.*

Proof. *Necessity.* Let $f : X \to Y$ be the identity function. Then f is continuous and *st.$\theta.\beta$.c.* by our hypothesis. For any open set U of X and any point x of U, we have $f(x) = x \in U$ and there exists $G \in \text{SPO}(X,x)$ such that $f(\text{spCl}(G) \subset$ U. Therefore, we have $x \in G \subset \text{SpCl}(G) \subset$ U. It follows from Theorem 2.1 of [2] that X is β-regular.

Sufficiency, Suppose that $f : X \to Y$ is continuous and X is β-regular. For any $x \in$ X and any open neighborhood V of $f(x)$, $f^{-1}(V)$ is an open set of X containing x. Since X is β-regular; there exists $U \in \text{SPO}(X)$ such that $x \in U \subset \text{SpCl}(U) \subset f^{-1}(V)$ by Theorem 2.1 of [2]. Therefore, we have $f(\text{spCl}(U)) \subset V$. This shows that f is *st.$\theta.\beta$.c.*

Theorem 3.6. *Let X be a semipre-regular space. Then $f : X \to Y$ is st.$\theta.\beta$.c. if and only if f is β-continuous.*

Proof. Suppose that f is β-continuous. Let $x \in X$ and V be any open set of Y containing $f(x)$. By the β-continuity of f, we have $f^{-1}(V) \in \text{SPO}(X,x)$ and hence there exists $U \in \text{SPO}(X,x)$ such that $\text{spCl}(U) \in f^{-1}(V)$. Therefore, we obtain $f(\text{spCl}(U)) \subset V$. This shows that f is *st.$\theta.\beta$.c.* The converse is obvious.

4. SOME PROPERTIES

Theorem 4.1. *Let $f : X \to Y$ be a function and $g : X \to X \times Y$ the graph function of f. Then, the following properties hold :*

(1) *If g is st.$\theta.\beta$.c., then f is st.$\theta.\beta$.c. and X is β-regular.*

(2) *If f is st.$\theta.\beta$.c. and X is semipre-regular, then g is st.$\theta.\beta$.c.*

Proof. (1) Suppose that g is *st.$\theta.\beta$.c.* First, we show that f is *st.$\theta.\beta$.c.* Let $x \in X$ and V be an open neighborhood of $f(x)$. Then $X \times V$ is an open set ot $X \times Y$ containing $g(x)$. Since g is *st.$\theta.\beta$.c.*, there exists $U \in SPO(X,x)$ such that $g(\text{spCl}(U)) \subset X \times V$. Therefore,

we obtain $f(\mathrm{spCl}(U)) \subset V$. Next, we show that X is β-regular. Let U be any open set of X and $x \in U$. Since $g(x) \in U \times Y$ and U × Y is open is X × Y, there exists G ∈ SPO (X,x) such that $g(\mathrm{spCl}(G)$ $U \times Y$. Therefore, we obtain $x \in G$ $\mathrm{spCl}(G) \subset U$ and hence X is β-regular by Theorem 2.1 of [2].

(2) Let $x \in X$ and W be any open set of $X \times Y$ containing $g(x)$. There exist open sets $U_1 \subset X$ and $V \subset Y$ such that $g(x) = (x,f(x)) \in U_1 \times V \subset W$. Since f is *st.θ.β.c.*, there exists $U_2 \in$ SPO (X,x) such that $f(\mathrm{spCl}(U_2) \subset V$. Since X is semipre-regular and $U_1 \cap U_2 \in \mathrm{SPO}(X,x)$, there exists $U \in$ SPO (X,x) such that $x \in U \subset \mathrm{spCl}(U) \subset U_1 \cap U_2$. Therefore, we obtain $g(\mathrm{spCl}(U)) \subset U_1 \times \mathrm{f}(\mathrm{spCl}(U_2)) \subset \mathrm{U}_1 \times \mathrm{V} \subset \mathrm{W}$. This shows that g is *st.θ.β.c.*

Corollary 4.2. *Let X be a semipre-regular space. Then, a function $f : X \to Y$ is st.θ.β.c. if and only if the graph function $g : X \to X \times Y$ is st.θ.β.c.*

Lemma 4.3. (Abd El-Monsef et. al. [1] *Let A and X_0 be subsets of a space X.*

(1) *If $A \in$* SPO (X) *and Y is α-open in X, then $A \cap Y \in$* SPO (Y).

(2) *If $A \in$* SPO (Y) *and $Y \in$* SPO (X), *then $A \in$* SPO (X).

Lemma 4.4. *Let X be a topological space and A, Y subsets of X such that $A \subset Y \subset X$ and Y is α-open in X. Then the following properties hold :*

(1) *$A \in \mathrm{SPO}(X)$ if and only $A \in$ SPO (X),*

(2) $\mathrm{spCl}(A) \cap Y = \mathrm{spCl}_Y(A)$, *where $\mathrm{spCl}_Y(A)$ denotes the semipre-closure of A in the subspace Y.*

Proof. (1) Let $A \in \mathrm{SPO}(Y)$. Since every α-open set is β-open, by Lemma 4.3, we have $A \in \mathrm{SPO}(X)$. Conversely, let $A \in \mathrm{SPO}(X)$. By Lemma 4.3, $A = A \cap Y \in \mathrm{SPO}(Y)$.

(2) Let $x \in \mathrm{spCl(A)} \cap \mathrm{Y}$ and $\mathrm{V} \in \mathrm{SPO}(Y,x)$. Then, by (1) $V \in$ SPO (X,x) and hence $\mathrm{V} \cap \mathrm{A} \neq \phi$. Therefore, $\mathrm{x} \in \mathrm{spCl}_Y(A)$. Conversely, let $\mathrm{x} \in \mathrm{spCl}_Y(A)$ and $V \in$ SPO (X,x). Then by Lemma 4.3 $x \in V \cap Y \in \mathrm{SPO(Y)}$ and hence $\phi \neq \mathrm{A} \cap (\mathrm{V} \cap \mathrm{Y}) \subset \mathrm{A} \cap \mathrm{V}$. Therefore, we obtain $x \in \mathrm{spCl(A)} \cap Y$.

Theorem 4.5. *If $f : X \to Y$ is st.θ.β.c. and X_0 is an α-open subset of X, then the restriction $f / X_0 : X_0 \to Y$ is st.θ.β.c.*

Proof. For any $x \in X_0$ and any open neighbourhood V of $f(x)$, there exists $U \in \mathrm{SPO}(X,x)$ such that $f(\mathrm{spCl}(U)) \subset V$ since f is *st.θ.β.c.* Put $U_0 = U \cap X_0$, then by Lemmas 4.3 and 4.4, $U_0 \in \mathrm{SPO}(X_0,x)$ and $\mathrm{sp}\,Cl_{X_0}(U_0) \subset \mathrm{spCl}(U_0)$. Therefore, we obtain

$$(f/X_0)(\mathrm{sp}\,Cl_{X_0}(U_0))) = f(sp\,Cl_{X_0}(U_0)) \subset f(\mathrm{spCl}(U_0)) \subset f(\mathrm{pCl}(U)) \subset \mathrm{V}.$$

This shows that f / X_0 is *st.θ.β.c.*

Definition 4.6. A function $f : X \to Y$ is said to be

(1) *β-irresolute [8] if $f^{-1}(V) \in SPO(X)$ for each $V \in SPO(Y)$,*

(2) *pre-β-open [8] if $f(U) \in SPO(Y)$ for each $U \in SPO(X)$.*

Lemma 4.7. *If $f : X \to Y$ is β-irresolute and V is an sp-θ-open in Y, then $f^{-1}(V)$ is sp-θ-open in X.*

Proof. Let V be an sp-θ-open set of Y and $x \in f^{-1}(V)$. There exists $W \in SPO(Y)$ such that $f(x) \in W \subset spCl(W) \subset V$. Since f is β-irresolute, we have $f^{-1}(W) \in SPO(X)$ and $f^{-1}(spCl(W)) \in SPC(X)$. Therefore, we obtain $x \in f^{-1}(W) \subset spCl(f^{-1}(W)) \subset f^{-1}(spCl(W)) \subset f^{-1}(V)$. This shows that $f^{-1}(V)$ is sp-θ-open in X.

Theorem 4.8. *Let $f : X \to Y$ and $g : Y \to Z$ be functions. Then, the following properties hold :*

(1) *If f is st.θ.β.c. and g is continuous, then the composition $g \circ f : X \to Y$ is st.θ.β.c.*

(2) *If f is β-irresolute and g is st.θ.β.c., then $g \circ f$ is st.θ.β.c.*

(3) *If $f : X \to Y$ is a pre-β-open bijection and $g \circ f : X \to Z$ is st.θ.β.c., then g is st.θ.β.c.*

Proof. (1) This is obvious from Theorem 3.1.

(2) This follows immediately from Theorem 3.1 and Lemma 4.7.

(3) Let W be any open set of Z. Since $g \circ f$ is *st.θ.β.c.*, $(g \circ f)^{-1}(W)$ *is sp-θ*-open in X. Since f is pre-β-open and bijective, f^{-1} is β-irresolute and by Lemma 4.7 we have $g^{-1}(W) = f((g \circ f)^{-1}(W)$ is sp-θ-open in Y. Hence, by Theorem 3.1 g is *st.θ.β.c.*

Let $\{X_\alpha : \alpha \in A\}$ be a family of topological spaces, A_α a nonempty subset of X_α for each $\alpha \in A$ And $X = \Pi\{X_\alpha : \alpha \in A\}$ denote the product space, where A is nonempty.

Lemma 4.9. (Abd El-Monsef [2] *Let n be a positive integer and*

$A = \prod_{j=1}^{n} A_{\alpha j} \times \prod_{\alpha \neq \alpha j} X_\alpha$. *Then the following properties hold :*

(1) $A \in$ SPO(X) if and only if $A_{\alpha_j} \in SPO(X_{\alpha_j})$ *for each j = 1, 2, ..., n.*

(2) $spCl\left(\prod_{\alpha \in A} A_\alpha\right) \subset \prod_{\alpha \in A} spCl(A_\alpha)$.

Theorem 4.10. *If a function* $f_\alpha : X_\alpha \to Y_\alpha$ *is st.θ.β.c. for each* $\alpha \in A$. *Then the product function* $f : \prod X_\alpha \to \prod Y_\alpha$, *defined by* $f(\{x_\alpha\}) = \{f_\alpha(x_\alpha)\}$ *for each* $x = \{x_\alpha\}$, *is st.θ.β.c*

Proof. Let $x = \{x_\alpha\} \in \prod X_\alpha$ and W be any open set of $\prod Y_\alpha$ containing $f(x)$. Then there exists an open set V_{α_j} of Y_{α_j} such that

$$f(x) = \{f_\alpha(x_\alpha)\} \in \prod_{j=1}^{n} V_{\alpha j} \times \prod_{\alpha \neq \alpha j} Y_\alpha \subset W.$$

Since f_α is *st.θ.β.c.* for each α, there exists $U_{\alpha j} \in SPO(X_{\alpha j}, X_{\alpha j})$ such that $f_{\alpha j}(spCl(U_{\alpha j})) \subset V_{\alpha j}$ for $j = 1, 2, \ldots, n$. Now, put $U = \prod_{j=1}^{n} U_{\alpha j} \times \prod_{\alpha \neq \alpha j} X_\alpha$. Then, it follows from Lemma 4.9 that U ∈ SPO $(\Pi X_\alpha, x)$. Moreover, we have

$$f(\mathrm{spCl}(U)) \subset f(\prod_{j=1}^{n} spCl(U_{\alpha j}) \times \prod_{\alpha \neq \alpha j} X_\alpha) \subset \prod_{j=1}^{n} f_{\alpha j}(spCl(U_{\alpha j})) \times \prod_{\alpha \neq \alpha j} Y_\alpha \subset$$

$$\prod_{j=1}^{J} V_{\alpha j} \times \prod_{\alpha \neq \alpha j} Y_\alpha \subset W$$

This shows that f is *st.θ.β.c.*

5. *St.θ β.c.* FUNCTIONS AND SEPARATIONS AXIOMS

A space X is said to be *sp-*T_2 or β-T_2 [8] if for each pair of distinct points x and y in X, there exist $U \in$ SPO(X,x) and $V \in$ SPO(X,y) such that $U \cap V = \Phi$.

Theorem 5.1. *If* $f : X \to Y$ *is a st.θ.β.c. injection and* Y *is* T_0, *then* X *is sp-*T_2.

Proof. Suppose that Y is T_0. Let x and y be any distinct points of X. Since f is injective, $f(x) \neq f(y)$ and there exists either an open neighbourhood V of $f(x)$ not containing $f(y)$ or an open neighbourhood W of $f(y)$ not containing $f(x)$. If the first case holds, then there exists U ∈ SPO(X,x) such that $f(\mathrm{spCl}(U)) \subset V$. Therefore, we obtain $f(y) \notin f(\mathrm{spCl}(U))$ and hence X—spCl$(U) \in$ SPO(X,y). If the second case holds, then we obtain the similar result. Therefore, X is *sp-*T_2.

Theorem 5.2. *If* $f : X \to Y$ *is a st.θ.β.c. function and* Y *is Hausdorff, then a subset* $E = \{(x,y) : f(x) = f(y)\}$ *is sp-θ-closed in* $X \times X$.

Proof. Suppose that $(x,y) \notin E$. It follows that $f(x) \neq f(y)$. Since Y is Hausdorff, there exist disjoint open sets V and W in Y containing $f(x)$ and $f(y)$, respectively. Since f is *st.θ.β c*, there exist $U \in$ SPO(X,x) and $G \in$ SPO(X,y) such that $f(\mathrm{spCl}(U)) \subset V$ and $f(\mathrm{spCl}(G)) \subset W$. Set $D = U \times G$. It follows that $(x,y) \in D \in$ SPO$(X \times X)$ and spCl$(D) \cap E \subset$ [spCl(U) × spCl(G)] $\cap E = \Phi$. Therefore, E is *sp-θ*-closed in $X \times X$.

For a function $f : X \to Y$ the subset $\{(x,f(x)) : x \in X\}$ of $X \times Y$ is called the *graph* of f and is denoted by $G(f)$.

Definition 5.3. The graph $G(f)$ of a function $f : X \to Y$ is said to be *strongly sp-closed* if for each $(x,y) \in (X \times Y) - G(f)$, there exist $U \in \mathrm{SPO}(X,x)$ and an open set V in Y containing y such that $(\mathrm{SpCl}(U)) \times V) \cap G(f) = \Phi$.

Lemma 5.4. *The graph $G(f)$ of a function $f : X \to Y$ is strongly sp-closed in $X \times Y$ if and only if for each point $(x,y) \in (X \times Y) - G(f)$, there exist $U \in SPO(X,x)$ and an open set V in Y containing y such that $f(spCl(U)) \cap V = \Phi$.*

Theorem 5.5. *If $f : X \to Y$ is st.θ.β.c. and Y is Hausdorff, then $G(f)$ is strongly sp-closed in $X \times Y$.*

Proof. Let $(x,y) \in (X \times Y) - \mathrm{G(f)}$. It follows that $f(x) \neq y$. Since Y is Hausdorff, there exist disjoint open sets V and W in Y containing $f(x)$ and y, respectively. Since f is *st.θ.β.c.*, there exists $U \in \mathrm{SPO}(X,x)$ such that $f(spCl(U)) \subset V$. Therefore, $f(spCl(U)) \cap W = \Phi$. and $G(f)$ is strongly *sp*-closed in $X \times Y$.

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Jour. Pure Math., Vol. 19, 2002, pp. 41-47

s–A RADICAL OF A NEAR-RING WITH CHAIN CONDITION

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ABSTRACT : This papaer deals with results on near-rings with ascending chain conditions on annihilators having no infinite direct sum of ideals (subgroups) and with parts satisfying the acc or the dcc on its substructures. The s-A radical of such parts gives the corresponding factor near-ring a similar structure. A countable s-A radical has no essential extension in N. A minimal ideal satisfying the acc on its left N-subgroups satisfies the descending chain conditions. In some cases the s-A radical coincides with the right annihilator of the left annihilator of the s-A radical coincides with the right annihilator of the left annihilator of the s-A radical. If the socle of N is with dcc on its N-subgroups than N also inherits the same character.

Mathematics Subject Classification (AMS 2000) : 16 Y 30, 16 P 40, 16 P 60

Key words and phrases : Near-rings, s-A radical, Essential N-subgroup, Socle, Annihilators, chain conditions.

1. INTRODUCTION

The study of near-rings with chain conditions is well developed [2,3,6,8,9] but the modifications on conditions such as ascending chain or descending chain on substructures of near-rings have left scopes for further studies. In our continuous afford we have studied near rings satisfying acc on annihilator[2,3,6]. The objective of this work is on near rings with acc on annihilators having no infinite direct sum of ideals(subgroups) and with parts satisfying the acc or the dcc on its substructures. The utility of such substructures motivates one to study the strictly Artinian radical (s A radical) of such a near-ring. Slicing a radical out from near-ring will yield a simpler and amenable near-ring. Further information about the original near-ring from the structure of the sliced radical can be also expected.

In [3] we introduced the concept of such a particular type of radical substructure, termed as stricIty Artinian radical. We define a strictly Artinian radical, the s A radical as the sum s-A(N) of all left ideals of N, each satisfying the dcc on its left N-subgroups. This paper deals with results on this radical character of a strongly semiprime near-ring N satisfying the conditions :

(i) N has no infinite independent family of left N- subgroups of it

and (ii) N satisfies the acc on left annihilators of subsets of N.

Here we prove the inheritance of dcc of a near-ring modulo the left annhilator of a countable s A radical. The s A radical has no countable extensions in N. If N is with acc on its left N-subgroups and s-A(N) is an invariant subnear-ring of N then s-A(N) = r(1(s-A(N))). If the socle of N is with dcc on its N-subgroups then N is also with dcc on its left N-subgroups.

2. DEFINITIONS AND NOTATIONS

All basic definitions used in the paper are referred to Pilz [9]. Throughout the paper N stands for a zero symmetric right dgnr with unity. E will denote a left N-group $_NE$. A left N-subgroup A of N will mean an N-subgroup of $_NN$ and left ideal of N will mean an idal of N will mean an ideal of $_NN$. Let A and B be two left N-subgroups of N, with A $\subseteq$ B then A is said to be an *essential left N-subgroup* of B, if any N-subgroup C ($\neq$ 0) of B has non zero intersection with A. If A is an essential left N-subgroup of B we say B is an *essential extension* of A in N. A left N-subgroup A of N is a *weakly essential left* N-*subgroup* of N if for any left ideal I ($\neq$ 0) of N, A $\cap$ I $\neq$ (0). It is clear that an essential left N-subgroup of N is a weakly essential left N-subgroup of it. But the converse is not true.

In the near-ring N = {0, a, b, c, x, y} defined as in (H) S^3, [9, p.342 (37)] we note that {0,a}, {0,b}, {0,c}, {0,x,y} are proper non zero left N-subgroups where {0,x,y} is a left ideal. This is not an essential left N-subgroup though it is weakly essential. A left ideal which is weakly essential as a left N-subgroup is an essential left ideal of N. Near-ring N is left singular if no non-zero element of N annihilates any essential left N-subgroup of N from the right. An element x $\in$ N is regular if there is an element y $\in$ N such that xyx = x and N is regular if each element of N is regular. Near-ring N is strongly semi prime if it has no nonzero nilpotent subset. It is easy to see that a regular near-ring is always strongly semiprime. And a partial converse of it is seen in Oswald [Corollary 4., [7]]. The collection of all maximal left annihilators of the type (P =) 1 (A), where A is a nonzero left N-subgroup of N is denoted by Γ. We confine our discussion to the subfamily P ($\subseteq \Gamma$) ={P | P = 1(A), where A is a nonzero invariant subnear-ring of N}. In the near-ring N = {0,a,b,c} under addition and multiplication defined as in [9, p.340(7)] the nonzero left N-subsets are {0,a}, {0,b}, {0,a,b} and N. Clearly their left annihilators are {0}, {0,a} and {0,b}. Thus Γ = {{0,a}, {0,b}}. On the other hand, nonzero invariant sub near-rings are {0,a}, {0,b}} and N and their left annihilators are {0}, {0,a}, {0,b}. Here *P* = {{0,a}, {0,b}}. Thus this example shows P $\subseteq \Gamma$. In near-ring N = {0,a,b,c} [[1], 2.1, 10], B = {0,b} is the only proper left ideals of N and clearly BN = NB = B. If N = {0,a,b,c} is a near-ring as defined in [[1],2.2,13] it is seen that A = {0,a}, B = {0,b} and C = {0,c} are proper left ideals of N. Clearly, AB = B, BA = (0), BC = (0), CB = B, AC = C, CA = A, A^2 = A, B^2 = (0) and C^2 = C.

The above examples show that any finite product of (left) ideals is an (left) ideal. In case of strongly regular near-ring we get that finite product of (left) ideals is an (left) ideal

as in [5]. In this sense such near-rings are termed as (left) *ideal closed near-rings*, in short (1,i) closed *near-rings*. Thus a strongly regular near-ring is (1,i) closed. But the near-rings given above are not strongly regular though the product of two ideals is again an ideal. So, an (1,i) closed near-ring need not be strongly regular. N is called (1,i) *closed* if any finite product of left ideals is also a left ideal [3]. We define *s A redical of* N as the sum s-A(N) of all left ideals of N, each satisfying the dcc on its left N-subgroups.

If N is with dcc on its left N-subgroups then s-A(N) = N and if N has non zero left ideals with dcc on its left N-subgroups then s-A(N) = (0).

$N = \begin{bmatrix} Z & Q \\ Q & Q \end{bmatrix}$ is a near-ring with acc on its left annihilators and it has no infinite direct sum of left ideals. If e_{ij} denotes the matrix with 1 in the (i,j)th position and 0's elsewhere, then the left ideals Qe_{21}, Qe_{22}, of N are with dcc on its N-subgroups (where Qe_{21}, Qe_{22}, denote the set of all 2X2 matrices with an arbitrary element of Q in the (1,2)th-position and (2,2)th-position respectively). Here s-A(N) = $Qe_{21} + Qe_{22}$.

Socle of E denoted by Soc(E) is the sum of all simple ideals of E. Soc(E) is also the intersection of all essential ideals of E [66]. N-group E is *finitely* N-*cogenerated* in case for every set of N-subgroups of E, $\cap A = 0$ implies $\cap F = 0$ for some finite $F \subseteq A$.

Now we note the following.

Note 2.1. If N is strongly semiprime near-ring with acc on left annihilators then N is left non singular.

Note 2.2. If N satisfies the acc on annihilators than $Z_l(N)$ is a nil invariant subset of N.

Note 2.3. If for some nonzero left N-subgroup A of a strongly semiprime near-ring N, P = l(A) is a maximal left annihilator then P is a minimal strongly prime ideal.

Note 2.4. If for some nonzero left N-subgroup A of a strongly semiprime near-ring N, P = l(A) is a maximal left annihilator then P is a minimal strongly prime ideal.

3. PRELIMINARIES

Lemma 3.1. If N satisfies the acc on left annihilators, then $Z_l(N)$ is nilpotent.

Proof. We write $Z = Z_l(N)$. As $Z \supseteq Z^2 \supseteq \ldots$, we get $l(Z) \subseteq l(Z^2) \ldots$ Since N satisfies the acc on left annihilators, we have a $t \in Z^+$ such that $l(Z^t) = l(Z^{t+1}) = \ldots$

If $Z^{t+1} \neq (0)$ then there is an element $a \in Z$ such that $aZ^t \neq (0)$.

We choose $l(a)$ as large as possible with $aZ^t \neq (0)$

Now $b \in Z$ gives $1(b) \subseteq_e N$ which implies $1(b) \cap Na \neq (0)$ and so $na \in 1(b)$, $na \neq 0$ for some $n \in N$. Z being invariant N-subset of N, we have $ab \in Z$. Also $1(a) \subseteq 1(ab)$.

Now $na \neq 0$ and $nab = 0$ gives $1(a) \subset 1(ab)$. So, the choice of a gives that $abZ^{t} = (0)$.

Thus, $aZ^{t+1} = (0)$, a contradiction and hence, $Z^{t+1} = (0)$ implies Z is nilpotent.

Lemma 3.2. If $A \subseteq_e E$, $B \subseteq_e E$ then $A \cap B \subseteq_e E$

Proof. Let D ($\neq 0$) be an N-subgroup of E. As $A \subseteq_e E$, $A \cap D \neq (0)$. Also $A \cap D$ is an N-subgroup of E and $B \subseteq_e E$ gives $(A \cap D) \cap B \neq (0)$ implies $(A \cap B) \cap D \neq (0)$. Thus $A \cap B \subseteq_e E$.

Lemma 3.3. If A, B are left N-subgroups of N with $A \subseteq B$ and A, B/A satisfy the dcc on its N-subgroups then B is also with dcc on its N-subgroups.

Proof. Let $B_1 \supseteq B_2 \supseteq \dots \dots \dots$ be a descending chain of N-subgroups of B. Since B/A is with dcc on its N-subgroups, there is an $r \in Z^+$ such that $B_r + A = B_{r+1} + A$. Also A is with dcc on its N-subgruops implies there is an integer $s \in Z^+$ such that $A \cap B_s = A \cap B_{s+1}$, for $A \cap B_1 \supseteq A \cap B_2 \supseteq \dots$ is a descending chain of N-subgroups of A. Now, if $t = \max(r,s)$ then $B_t = B_t \cap (B_t + A)$ implies $B_t \cap (B_{t+1} + A) = B_{t+1} + (B_t \cap A) = B_{t+1} + (B_{t+1} \cap A) = B_{t+1}$. Thus, B is with dcc on its N-subgroups.

Lemma 3.4. If I is a finitely generated N-group and N is with dcc on its left N-subgroups then I is with dcc on its N-subgroups.

Proof. As I is finitely generated, we have finite set $\{i_1, i_2, \dots I_t\} \subseteq I$, such that

$Ni_1 \oplus \dots \oplus Ni_t \to I \to 0$ is exact. Since N is with dcc on its N-subgroups, $N^F = \oplus N$ is also so. As I is isomorphic to a factor of N^F we get I is with dcc on its N-subgroups.

Lemma 3.5. If N-group E is with dcc on its N-subgroups then E is finitely N-cogenerated.

Proof. Let A be the collection of N-subgroups of E with dcc on its N-subgroups. If $\cap A = 0$ and $B = \{ \cap F \mid F$ is a finite sub collection of A$\}$ then B has a minimal element. Since $\cap A = 0$, it follows that this minimal element must be zero. So for some finite subclass F of A, $\cap F = 0$. Thus the result follows.

Lemma 3.6. If N-group E is with dcc on its N-subgroups then Soc E is essential in E.

Proof. Let B be an ideal of E with $(\mathrm{Soc}E) \cap B = 0$. Now Soc E = intersection of all essential ideals of E. Since E is finitely cogenerated by 3.5. So, there exists essential ideals $I_1, I_2, \dots I_k$ of E with $I_1 \cap I_2 \cap \dots \cap I_k \cap B = 0$. Also $I_1, I_2, \dots I_k$ are essential in E. By 3.1.9., $I_1 \cap I_2 \cap \dots \cap I_k \subseteq_e E$. Thus $B = 0$ giving thereby Soc $E \subseteq_e E$.

As in ring theory we get the following Lemma.

Lemma 3.7. If N-group E is with dcc on its N-subgroups and L is an N-subgroup of E then SocL = L $\cap$ SoeE.

Lemma 3.8. If I is a left ideal of a strongly semiprime near-ring N and if N/1(I) (= $\overline{N}$) satisfies the dcc on its left $\overline{N}$-subgroups then I also satisfies the dcc on its N-subgroups.

Proof. If $A_1 \supseteq A_2 \supseteq \ldots$ is a descending chain of N-subgroups of I then we get a descending chain $Q_1 \supseteq Q_2 \supseteq \ldots$ of left $\overline{N}$-subgroups of $\overline{N}$ where $Q_i = \{a_i + 1(I) \mid a_i \in A_i\}$. As $\overline{N}$ is with dcc on its left $\overline{N}$-subgroups so, $Q_t = Q_{t+1}$, for some $t \in Z^+$. Now $A_i \cap 1(I) \subseteq A_i$ and 1(I) implies $A_i \cap 1(I) \subseteq I$. So $[(A_i \cap 1(I)]^2 \subseteq 1(I)I = (0)$ which gives $[A_i \cap 1(I)]^2 = (0)$. And N being strongly semiprime $A_i \cap 1(I) = (0)$. Thus for all i, $A_i \cap 1(I) = (0)$. So, $Q_t = Q_{t+1} \Rightarrow A_t = A_{t+1} = \ldots$ The result follows.

For two left ideals A, B of N, (A + B) / B and A / (A $\cap$ B) are isomorphic. If A, B are with dcc on their N-subgroups A $\cap$ B is also so. Thus (A + B) / B is also with dcc on its N-subgroups. Hence A + B is with dcc on its N-subgroups. Thus we easily get.

Lemma 3.9. [3] s-A(N) is a left ideal of N and it satisfies the dcc on its N-subgroups.

In case of $N = \begin{bmatrix} Z & F \\ 0 & Z \end{bmatrix}$, $F = Z/2Z$, it satisfies the condition that for ideals I and X, if N / X is with dcc on N-subgroups then there is an ideal Y such that N / Y is also is so.

The following result applies to near-ring N which satisfies the property that if N / X is with dcc on its N-subgroup then so is N / Y. The property that N is a (1,ı) closed near-ring plays an important role.

Lemma 3.10. [3] If N is a strongly semiprime near-ring as above and is with distributively generated left annihilators, s-A(N) is minimal and countable invariant sub near-ring then for any minimal strongly prime ideals of N with dcc on its left N-subgrups, s-A(N) + 1(s-A(N)) contains a non zero divisor.

4. MAIN RESULTS

We now prove our main results on s-A radical of a strongly semiprime near-ring.

Theorem 4.1. Let N be a strongly semiprime near-ring. If s-A(N) is countable then s-A(N) has no countable essential extension in N.

Proof. Let B be a proper countable left N-subgroup of N such that s-A(N) is an essential left N-subgroup of B. So B has an essential left N-subgoup which satisfies the dcc on its left N-subgroups. Since B is countable, by Lemma 3.1[6] there exists a finite $S = \{y_1, \ldots Y_t\}$ ($\subseteq$B) such that 1(B) = 1(S).

We define $\varphi : N/1(B) \to Ny_1 \oplus \ldots\ldots\oplus Ny_t$ such that $\varphi(\bar{x}) = (xy_1, \ldots xy_t)$. For any $\bar{a}, \bar{b} \in N/1(B)$ we get, $\varphi(\bar{a} + \bar{b}) = \varphi(\bar{a}) + \phi(\bar{b})$ and $\varphi(n\bar{a}) = n\varphi(\bar{a})$, $n \in N$. Also $\varphi(\bar{a}) = \varphi(\bar{b})$ gives $(ay_1, \ldots, ay_t) = (by_1, \ldots, by_t)$ and so $(a-b)y_i = 0$. Hence, $(a-b) \in 1(y_i)$, for $i = 1, \ldots, t$ which implies $a-b \in 1(y_1) \cap \ldots \cap 1(y_t) = 1(B)$. Thus, $\bar{a} = \bar{b}$. And so φ being an N-monomorphism, $N/1(B))$ can be embedded in $Ny_1 \oplus \ldots\ldots\oplus Ny_t$ as an N-group and thus $N/1(B)$ is embedded in the direct sum of a finite number of copies of B. Also $N/1(B)$ has an essential left N-subgroup $(s\text{-}A(N) \oplus 1(B)) / 1(B)$ which satisfies the dcc on its left N-subgroups, as s-A(N) is with dcc. Thus $N/1(B)$ has an essential left subgroup with dcc on its left N-subgroups. By Lemma 3.6 and 3.7, $M = \mathrm{Soc}\,(N/1(B))$ is an essential left N-subgroups of $N/1(B)$. If $x \in 1(M)$, for some $x \in N/1(B)$ then $xM = (0)$. Since N is strongly semiprime, $1(M) \subseteq r(M)$. Thus $xM = Mx = 0$ implies $x \in Z_1(N/1(B))$ and so $1(M) \subseteq Z_1(N/1(B))$.

If $\mathcal{N}$ is the nil radical of $N/1(B)$ then it is the largest nil ideal which gives $Z_1(N/1(B)) \subseteq \mathcal{N}$ and thus $1(M) \subseteq \mathcal{N}$. By what we have got, M is with dcc on its left $N/1(B)$ $(= \bar{N})$-subgroups. So $N/1(M)$ is with dcc on its $\bar{N}$-subgroups and thus $\bar{N}/\mathcal{N}$ is with dcc on its $\bar{N}$-subgroups. So by Lemma 3.8, B satisfies the dcc on its left N-subgroups. Therefore $B \subseteq s\text{-}A(N)$ which gives $s\text{-}A(N) = B$, a contradiction. Thus, s-A(N) has no countable essential extension in N.

Theorem 4.2. Let N be a strongly semiprime near-ring. If s-A(N) is countable then the N-group $N/1(s\text{-}A(N))$ satisfies the dcc on its N-subgroups.

Proof. s-A(N) being countable, by Lemma 3.1[6], there is a finite set $S = \{y_1, \ldots y_t\}$ s-A(N) such that $1(s\text{-}A(N)) = 1(y_1) \cap \ldots \cap 1(y_t) = 1(S)$. As in the above theorem $N/1(s\text{-}A(N))$ can be embedded in $Ny_1 \oplus \ldots\ldots\oplus Ny_t$ as an N-group. Thus $N/1(s\text{-}A(N))$ satisfies the dcc on its N-subgroups.

Theorem 4.3. If N is as above then $s\text{-}A(N/s\text{-}A(N)) = (\bar{0})$.

Proof. Let I be a left ideal of N each such that s-A(N) I and $I/s\text{-}A(N)$ be with dcc on its left $(N/s\text{-}A(N) =)\ \bar{N}$ subgroups. If $J/s\text{-}A(N)$ is an $\bar{N}$-subgroup of $I/s\text{-}A(N)$ then it is an N-subgroup of $I/s\text{-}A(N)$. Hence $I/s\text{-}A(N)$ is with dcc on its left N-subgroups. By Lemma 3.9, s-A(N) is with dcc on its left N-subgroups. Thus by Lemma 3.3, I satisfies the dcc on its N-subgroups. So, $I \subseteq s\text{-}A(N)$ which gives $I = s\text{-}A(N)$. Therefore $N/s\text{-}A(N)$ does not have any nonzero left ideal with dcc on its N-subgroups. This implies $s\text{-}A(N/s\text{-}A(N)) = (\bar{0})$.

Theorem 4.4. Let N be a strongly semiprime near-ring. If s-A(N) is an invariant sub near-ring of N then $s\text{-}A(N) = r(1(s\text{-}A(N)))$.

Proof. We denote s-A(N) by K. By theorem 4.2 $N/1(K)$ satisfies the dcc on its N-subgroups. We consider the map $N/1(K) \times r(1(K)) \to r(1(K))$ by $\bar{x}, \alpha \to x\alpha$. The map is well defined. For if $\bar{x} = \bar{y}$ then $x-y \in 1(K)$. Now $\alpha \in r(1(K))$ gives $1(K)\alpha = (0)$ and so $x\alpha = y\alpha$. Also $(\bar{x}+\bar{y})\alpha = \bar{x}\alpha + \bar{y}\alpha$, $(\bar{x}\ \bar{y})\alpha = \bar{x}\ (\bar{y}\alpha)$ and $\bar{1}\alpha = \alpha$ So, r(1(K)) is

a left $N/l(K)$ group. As N satisfies the acc on its N-subgroups, $r(l(K))$ is with dcc on its $N/l(K)$-subgroups. So, $r(l(K))$ satisfies the dcc on its N-subgroups. Thus $r(l(K)) \subseteq K$. Also if $x \in K$ then $l(K)\, x = (0)$ implies $x \in r(l(K))$. So, $K \subseteq r(l(K))$. Thus $K = r(l(K))$.

Theorem 4.5. Let N be a strongly semiprime near-ring as in Lemma 3.10. If Soc(N) is essential in N and Soc(N) is with dcc on its N-subgroups then N is also with dcc on its N-subgroups.

Proof. We set K = s-A(N). By lemma 3.10, $K + l(K)$ contains a non-zero divisor c. Using Lemma 3.2 [6] we get Nc is an essential left N-subgroup of N. If $x \in r(Nc)$ then $Ncx = 0$. Since N is strongly semiprime with acc on annihilators, by Note 2.1, $Z_l(N) = 0$. So, Nc being essential subgoup of N we get $x = 0$. Thus, $r(Nc) = (0)$. Also, $Nc \subseteq K + l(K)$ gives $r(K + l(K)) \subseteq r(Nc) = (0)$ and so $r(K + l(K)) = (0)$. Since, $r(K) \cap r(l(K)) \subseteq r(K + l(K)) = (0)$. So we get $r(K) \cap r(l(K)) = (0)$. Hence by Theorem 4.4, $r(K) \cap K = (0)$. Since Soc (N) is with dcc on its N-subgroups and Soc(N) is an essential left ideal of N we get $Soc(N) \subseteq K$ and thus $Soc(N) \cap r(K) = 0$. Hence $r(K) = (0)$. Now, $K = r(l(K)) = r(r(K)) = r(0) = N$. Thus, N is with dcc on its left N-subgroups.

ACKNOWLEDGEMENT

The author is thankful to Prof. K. C. Chowdhury, Department of Mathematics of Gauhati University for his suggestion and help in preparing this manuscript.

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Jour. Pure Math., Vol. 19, 2002, pp. 49-59

INTEGRATION OF A MORE GENERAL CLASS OF FUNCTION ON A TOPOLOGICAL SPACE

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ABSTRACT : Applying the concepts of regular θ-Baire measure [2] and regular θ-Borel measure [3], we here introduce the concept of integration of θ-Baire functions [2] and θ-Borel functions [3] with H-set support and finally we give an analogue but a generalized version of the famous Riesz-Markoff represantation theorem.

2000 AMS Subject Classification : 28A25

Key words and phrases : Locally θ-H-closed space [5], θ-CR-space [5], θ-T_2-space [5], θ-open set [6], θ-closed set [6], H-set [6], θ-Baire set [2], θ-Borel set [3], Continuous function with H-set support [2], Simple θ-Baire function [2], θ-Baire measure [2], θ-Borel measure [3], θ-content [3], regular θ-content [3].

1. INTRODUCTION

The theory of integration can be developed on two types of topological spaces, viz., locally compact T_2-space and complete metric space. We know that every continuous function with compact support is integrable on a locally compact T_2-space with respect to a regular Baire measure or regular Boarel measure. In [5] a class of topological spaces is innovated viz., locally θ-H closed, θ-CR, θ-T_2 space which is more general than a locally compact T_2-space in a non-regular space; also in [3], it is shown that on a locally θ-H-closed, θ-CR, θ-T_2 space, there always exists a regular θ-Borel measure. Using all these concepts, we here introduce the concept of integration of a continuous function with H-set support with respect to a regular θ-Baire measure or θ-Borel measure on a locally θ-H-closed, θ-CR, θ-T_2 space as is done by first integrating characteristic functions and subsequently, simple functions, non-negative *-measurable functions, then *-measurable functions in general (*—denoting either Baire or Borel).

Theorem 4.13. [3], which ascertain the possibility of extending a θ-Baire measure ν on a locally θ-H-closed, θ-CR, θ-T_2 space X to a unique regular θ-Borel measure μ might

* The Research is supported by a Minor Research Project of the UGC, India.

however arouse an atom of ambiguity which requires a solution at the very outset for the sake of consistency and this shall be done in Theorem 1.1 of next article.

However throughout the whole paper we would be primarily interested in the relationship between a regular θ-Borel measure and the integral which it assigns to continuous functions with H-set support.

Unless otherwise stated X shall denote a locally θ-H-closed, θ-CR, θ-T_2 space and $\mathcal{H}$ shall denote the class of all continuous functions on X with H-set support.

PREREQUISITES

0.1. Let A be an H-set and U be a θ-open set in X with $A \subset U$ ensure the existence of some $f \in \mathcal{H}$ such that $f = 0$ on $X \setminus U$, $f = 1$ on A and $0 \leq f \leq 1$ [2].

0.2. The class of all θ-Baire functions is the smallest class of real-valued continuous functions on X which contains $\mathcal{H}$ and is closed under sequential pointwise limit [2].

0.3. If f is integrable, g is measureable and $f = g$ a.e. then g is also integrable and $\int f d\mu = \int g d\mu$[4].

0.4. If f is measurable, g is integrable and $|f| \leq |g|$ a.e., then f is integrable [4].

1. INTEGRATION OF CONTINUOUS FUNCTIONS WITH H-SET SUPPORT

Theorem 1.1. *Let μ be a θ-Borel measure on X, and ν the θ-Baire restriction of μ. If f is a θ-Baire function, then f is μ-integrable if and only if it is ν-integrable, and in this case* $\int f d\mu = \int f d\nu$.

Proof. As usual, there shall be no loss of generality if we consider $f \geq 0$; with the usual practice we can select a sequence of simple θ-Baire functions such that $0 \leq f_n \uparrow f$ ......(1) ; of course, they are θ-Borel functions too. If $N(f)$ denotes the set $\{x \in X : f(x) \neq 0\}$ then evidently $\mu(N(f_n)) = \nu(N(f_n))$ and it is clear that f_n is μ-integrable if and only if it is ν-integrable, and that in this case we have $\int f_n d\mu = \int f_n d\nu$; in view of (1), it is now assured, by simple measure-theoretic arguments that f is μ-integrable if and only if it is ν-integrable and that in this case the values of the two integrals are equal.

Example 1.2. Let X be the real line and if τ_1 is the Euclidean topology on X and τ_2 is the topology of countable complements on X, we define τ to be the smallest topology generated by $\tau_1 \cup \tau_2$. Let $[a,b]$ be any closed subset of (X, τ_1). Then $[a,b]$ is a compact subset of (X,τ_1), also $[a,b]$ is an H-set but not compact subset of (X,τ). Hence any real valued

continuous function with support $[a,b]$ in (X,τ_1) is a continuous function with H-set support $[a,b]$ in (X,τ) but not with compact support.

Note 1.3. The class $\mathcal{H}$ of real valued continuous functions on X with H-set support is wider than the class of real valued continuous functions on X with with compact set support if X is non-regular.

Theorem 1.4. *If f be a real valued continuous function with H-set support on X, then f is ν-integrable, where ν is the θ-Baire measure.*

Proof. Let C be the H-set such that $f = 0$ on $X \setminus C$. First we prove that f is bounded on C. Infact, $f(C)$ is an H-set in $\mathbb{R}$, Since $\mathbb{R}$ is regular $f(C)$ is compact set in $\mathbb{R}$. Thus $f(C)$ is a bounded set in $\mathbb{R}$ and so f is a bounded function. Let α be an upper bound of f on C, then clearly $|f(x)| \leq \alpha\chi_c$; also $\alpha\chi_c$ is integrable and f is a θ-Baire function [2] hence f is ν-measurable [2], so f is ν-integrable on X (by 0.4). This completes the theorem.

Note 1.5. It follows from Theorem 1.4 that if μ is a θ-Borel measure, and ν is the θ-Baire restriction of μ, then every continuous function with H-set support is ν-integrable and hence μ-integrable by Theorem 1.1.

With $\mathcal{H}$—denoting the class of continuous functions with H-set support we can now assure a property common to θ-Baire measure and regular θ-Borel measure. Infact, if any two θ-Baire measure assign the same integral to each f in H, then they are identical ; the same is true for two θ-Borel measures as well.

Theorem 1.6. *If ν_1 and ν_2 are θ-Baire measure on X such that $\int f d\nu_1 = \int f d\nu_2$ for every f in $\mathcal{H}$, then $\nu_1 = \nu_2$*

Proof. Let D be any arbitrary $H\text{-}G_\delta^\theta$ set, then χ_D is a simple θ-Baire function and there exists a sequence of functions $\{f_n\}$ in $\mathcal{H}$ such that $f_n \downarrow \chi_D$ [2], this implies that $\int f_n d\nu \downarrow \nu(D)$, for every θ-Baire measure ν. Hence $\int f_n d\nu_1 \downarrow \nu_1(D)$ and $\int f_n d\nu_2 \downarrow \nu_2(D)$. But, according to assumption $\int f_n d\nu_1 = \int f_n d\nu_2$, for every n. So $\nu_1(D) = \nu_2(D)$ for every $H\text{-}G_\delta^\theta$ set i.e., $\nu_1 = \nu_2$ [2].

Theorem 1.7. *If μ_1 and μ_2 are regular θ-Borel measures on X such that $\int f d\mu_1 = \int f d\mu_2$ for every f in $\mathcal{H}$, then $\mu_1 = \mu_2$*

Proof. Let ν_i be the θ-Baire restriction of μ_i, $i = 1, 2$; by Theorems 1.1 and 1.4 we have $\int f d\nu_1 = \int f d\nu_2$ for every f in $\mathcal{H}$; hence $\nu_1 = \nu_2$ by Theorem 1.6. So it follows that $\mu_1(E) = \mu_2(E)$ for every θ-Borel set E[3]. Hence $\mu_1 = \mu_2$.

Lemma 1.8. *If f is any integrable θ-Baire function, then there exists a sequence f_n of integrable simple θ-Baire functions such that $f_n \to f$ pointwise and $\int (f - f_n)dv \to 0$, where v denotes the given θ-Baire measure.*

Proof. If suffices to consider the case when $f \geq 0$. By definition of integrability, there exists a sequence f_n of integrable simple functions such that $0 \leq f_n \uparrow f$, and $\int f dv$ is defined to be sup $\int f_n\, dv$. Thus $\int f_n\, dv \uparrow \int f dv$; since $f - f_n \geq 0$ we have $\int (f - f_n)dv$ $= \int f dv - \int f_n\, dv \to 0$

Theorem 1.9. *Let v be θ-Baire measure on X and suppose f is a v-integrable θ-Baire function. Then for every $\varepsilon > 0$, there exists a function g, given by $g = \sum_{j=1}^{m} a_j \chi_{A_j}$, where a_j's are real numbers and A_j's are mutually disjoint H-G_σ^θ sets such that $\int |f - g|\, dv \leq \varepsilon$.*

Proof. Without any loss of generality, by Lemma 1.8 we may assume $f = \sum_{1}^{m} \beta_j \chi_{B_j}$, where β_j's are real numbers and B_j's are mutually disjoint θ-Baire sets and $v(B_j) < +\infty$, for all j. If K is maximum of the finite set $\{|\beta_j| : 1 \leq j \leq m\}$ then clearly $|f| \leq K$.

Now, v being regular is inner regular, so for each B_j, there exists H-G_σ^θ set A_j such that $A_j \subset B_j$ and $v(B_j - A_j) \leq \dfrac{\varepsilon}{mK}$. Now, we define a function g = $\sum_{j=1}^{m} \beta_j \chi_{A_j}$.

Then $$f - g = \sum_{j=1}^{m} \beta_j \left(\chi_{B_j} - \chi_{A_j}\right) = \sum_{j=1}^{m} \beta_j \chi_{B_j - A_j}$$

So $$\int |f - g| \leq \sum_{j=1}^{m} |\beta_j| v\left(B_j - A_j\right) \leq \sum_{j=1}^{m} K \frac{\varepsilon}{mK} = \varepsilon$$

This Completes the theorem.

Our next theorem is an approximation of v-integrable function by a function in $\mathcal{H}$.

Theorem 1.10. *If f is any v-integrable function, where v is any θ-Baire measure on X, then for any $\varepsilon > 0$, there exists a function $g \in \mathcal{H}$ such that $\int |f - g|\, dv < \varepsilon$.*

Proof. In view of Theorem 1.9, we can assume $f = \sum_{j=1}^{m} \beta_j \chi_{A_j}$, where β_j's are real numbers and A_j's are mutually disjoint $H\text{-}G_\delta^\theta$ sets. Now corresponding to each χ_{A_j} (where A_j is an $H\text{-}G_\delta^\theta$ set) there exists a sequence $\{f_n^j\}$ of functions from $\mathcal{H}$ converging pointwise to χ_{A_j} [2]. Since linear combination of finite number of functions in $\mathcal{H}$ is again in $\mathcal{H}$, so finite linear combination of f_n^j's for j = 1, 2, ..., m is a sequence of functions say $\{f_n\}$ in $\mathcal{H}$ converging pointwise to f. Hence $\left\{\int f_n dv\right\}$ converges pointwise to $\int f dv$. So, for each ε > 0, there exists a continuous function with H-set support from $\{f_n\}$ say, $f_k = g$ such that $\int |f - g| \, dv < \varepsilon$. This completes the proof.

2. APPROXIMATION OF μ-INTEGRABLE FUNCTION BY A FUNCTION IN $\mathcal{H}$

We had studied extensively θ-Borel and θ-Baire functions along with θ-Borel and θ-Baire measures in [2] and [3]. The set of θ-Borel functions obviously contains the set of θ-Baire functions which is why it becomes quite pertinent to ask whether θ-Borel functions can be approximated by θ-Baire functions. The answer is not only in the affirmative as can be seen from the subsequent discussion, but also, even more, every μ-integrable function where μ is a regular θ-Borel measure, can be approximated by a function in $\mathcal{H}$.

Proposition 2.1. *Let μ be a regular θ-Borel measure and E is any θ-Borel set, then there exists a θ-Baire set F such that $\chi_E = \chi_F$ a.e. $[\mu]$.*

Proof. It is sufficient to prove that for each θ-Borel set E, there exists a θ-Baire set F such that $\mu(E \,\Delta\, F) = 0$.

Case I : Let $\mu(E) < +\infty$. By regularity, there exists a sequence of H-sets $\{C_n\}$ such that $C_n \subseteq$ E and $\mu(C_n) \uparrow \mu$(E). Let $G = \bigcup_{n=1}^{\infty} C_n$ then clearly $G \subset E$ and $\mu(G) = \mu(E)$ i.e., $\mu(E \setminus G) = 0$. Using regularity of μ, for each n; we may choose an $H\text{-}G_\delta^\theta$ set D_n such that $C_n \subset D_n$ and $\mu(D_n - C_n) = 0$ [3]; if we take $F = \bigcup_{n=1}^{\infty} D_n$, then F is a θ-Baire set such that $G \subset F$ and $\mu(F - G) = 0$; so $\mu(E \,\Delta\, F) = 0$.

Case II. $\mu(E) = \infty$; then $E \subset \bigcup_{i=1}^{\infty} C_i$ where C_i's are H-sets ; since $\mu(C_i) < +\infty$ for each i, μ is σ-finite ; thus there exists a sequence of θ-Borel sets $\{E_i\}$ of finite measures

such that $E = \bigcup_{i=1}^{\infty} E_i$. Then by case I; for each n; there exists a θ-Baire set F_n such that

$\mu(E_n \Delta F_n) = 0$; then if we take $F = \bigcup_{n=1}^{\infty} F_n$ clearly $\mu(E \Delta F) = 0$, since $E \Delta F \subset \bigcup_{n=1}^{\infty} E_n \Delta F_n$;

obviously, F is a θ-Baire set.

Theorem 2.2. *If f is any θ-Borel function and μ is a regular θ-Borel measure on X, then there exists a θ-Baire function h such that $f = h$ a.e.*

Proof. Here f is any θ-Borel function, so there exists a sequence $\{f_n\}$ of simple θ-Borel functions converging pointwise to f. Hence by Proposition 2.1, there exists a sequence of simple θ-Baire functions $\{g_n\}$ such that $g_n = f_n$ a.e. $[\mu]$.

So $g_n \to f$ a.e. $[\mu]$. Let E be the set such that $\mu(E) = 0$ and $g_n \to f$ pointwise on $X \setminus E$.

Let us construct the set F_{mn}, by

$$F_{mn} = \bigcup_{i,j \geq n} \left\{ x : |g_i(x) - g_j(x)| \geq \frac{1}{m} \right\};$$

then clearly F_{mn} is a θ-Baire set for each m, n [2]. Consequently,

$$F = \bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} F_{mn}$$

is a θ-Baire set. Now, since g_n is convergent on $X \setminus E$, so $F \subset E$ and so $\mu(F) = 0$.

If we define $h_n = \chi_{X \setminus F} g_n$; then h_n is a θ-Baire function [2] and $h_n = g_n$ a.e. $[\mu]$ and hence $h_n \to f$ a.e. $[\mu]$. Then h, the pointwise limit of $\{h_n\}$ is a θ-Baire function such that $f = h$ a.e. $[\mu]$.

Finally, we prove that, each μ-integrable θ-Borel function can be approximated by functions in $\mathcal{H}$, i.e., corresponding to each μ-integrable θ-Borel function f, there exists a sequence $\{f_n\}$ of functions in $\mathcal{H}$ converging pointwise to f and in this case $\int f \, d\mu = \lim_n \int f_n \, d\mu$.

Theorem 2.3. *Let μ be a regular θ-Borel measure X and suppose f is a μ-integrable θ-Borel function, then for every $\varepsilon > 0$, there exists a function $h \in \mathcal{H}$ such that $\int |f - h| \, d\mu \leq \varepsilon$.*

Proof. By Theorem 2.2, there exists a θ-Baire function g such that $g = f$ a.e. $[\mu]$, then g is μ-integrable (by 0.4). Let ν be the θ-Baire restriction of μ, then g is ν-integrable [by Theorem 1.1], then for any $\varepsilon > 0$, there exists a function $h \in \mathcal{H}$ such that

$$\int |g-h|\, dv \le \varepsilon \qquad \text{[by 1.10]}$$

$$\Rightarrow \quad \int |g-h|\, d\mu \le \varepsilon \qquad \text{[by 1.1]}$$

$$\Rightarrow \quad \int |f-h|\, d\mu \le \varepsilon \qquad (\text{since } |f-h| \le |g-h| \text{ a.e. } [\mu])$$

This completes the theorem.

3. GENERALIZED REPRESENTATION THEOREM

Notation 3.1. If f is any real valued function on X and $E \subset X$, then we write $f \geq E$ if $f(x) \geq 1$ for all $x \in E$ and $f(x) \geq 0$ if $x \in X \setminus E$.

Theorem 3.2. *If F is a positive linear form on $\mathcal{H}$, then there exists a unique regular θ-Borel measure μ such that $F(f) = \int f\, d\mu$, for all $f \in \mathcal{H}$.*

Proof. We define a set function

$$\lambda(H) \qquad = \inf\{F(f) : f \geq H \text{ and } f \in H\} \text{ for every } H\text{-set } H.$$

Since F is positive, $\lambda(H) \geq 0$ for every H-set H. Let H be any H-set and V be any θ-bounded θ-open set containing H. Then there exists a function $f \in \mathcal{H}$ such that $f(x) = 1$ for all $x \in H$ and $f(X \setminus V) = 0$ and $0 \leq f \leq 1$ (by 0.1). So $\lambda(H) \leq F(f) < \infty$. Therefore λ is finite.

To prove λ is monotone, let H_1 and H_2 are H-sets such that $H_1 \subset H_2$ and $f \geq H_2$, where $f \in \mathcal{H}$ (existence of such function is ensured by 0.1), then $f \geq H_1$ also and hence $F(f) \geq \lambda(H_1)$. Therefore $\inf\{F(f) : f \geq H_2\} \geq \lambda(H_1)$, i.e. $\lambda(H_2) \geq \lambda(H_1)$.

To prove λ is subadditive, let H_1 and H_2 be two H-sets, then if $f_1, f_2 \in \mathcal{H}$ be such that $f_1 \geq H_1$ and $f_2 \geq H_2$ (existence of such function is ensured by 0.1). Then $f_1 + f_2 \geq H_1 \cup H_2$.Clearly $f_1 + f_2 \in \mathcal{H}$, so $F(f_1) + F(f_2) = F(f_1 + f_2) \geq \lambda(H_1 \cup H_2)$.

Then clearly

$$\lambda(H_1) + \lambda(H_2) = \inf F(f_1) + \inf F(f_2) \geq \lambda(H_1 \cup H_2).$$

Using subadditivity, we shall prove that λ is additive. Suppose H_1 and H_2 are disjoint H-sets, then as X is θ-T_2, there exist θ-bounded θ-open sets $W_i \supset H_i$ such that $W_1 \cap W_2 = \phi$. So there exists $f_i \in \mathcal{H}$, for $i = 1, 2$ such that $f_i(H_i) = 1$ and $f_i(X \setminus W_i) = 0$ and $0 \leq f_i \leq 1$ (By 0.1). If $g \in \mathcal{H}$ be such that $g \geq H_1 \cup H_2$, then

$$\sum_{i=1}^{2} \lambda(H_i) \le \sum_{i=1}^{2} F(gf_i) = F\left(g \sum_{i=1}^{2} f_i\right) \le F(g)$$

and hence

$$\lambda(H_1) + \lambda(H_2) \leq \inf\{F(g) : g \geq H_1 \cup H_2\}$$

i.e., $$\lambda(H_1) + \lambda(H_2) \leq \lambda(H_1 \cup H_2).$$

Using subadditivity, additivity of λ follows. Therefore λ is a θ-content.

Next we shall show that λ is a regular θ-content. To prove this, let H be an H-set and $\varepsilon > 0$, then by definition of λ there exists a function $f \in \mathcal{H}$ such taht $f \geq H$ and

$$F(f) \leq \lambda(H) + \frac{\varepsilon}{2}.$$

If α is a real number, $0 < \alpha < 1$ and $K = f^{-1}[\alpha, \infty)$, then

$$H \subset f^{-1}[1,\infty) \subset f^{-1}(\alpha,\infty) \subset f^{-1}[\alpha,\infty) = K.$$

So K is an H-set [2] and since f is real valued continuous function, hence θ-continuous ; so $f^{-1}(\alpha,\infty)$ is θ-open [since (α, ∞) is θ-open]. Clearly, $\frac{1}{\alpha} f \geq K$ and $\frac{1}{\alpha} f \in \mathcal{H}$ also $\frac{1}{\alpha} f(x) \geq 1$ for all $x \in K$ and $\frac{1}{\alpha} f(x) = 0$ when $x \in X \setminus K$.

Now, $$\lambda(K) \leq F\left(\frac{1}{\alpha} f\right) = \frac{1}{\alpha} F(f) \leq \frac{1}{\alpha}\left\{\lambda(H) + \frac{\varepsilon}{2}\right\}.$$

By choosing α near enough to 1 we can write $\frac{1}{\alpha}\left(\lambda(H) + \frac{\varepsilon}{2}\right) \leq \lambda(H) + \varepsilon$. Therefore $\lambda(K) \leq \lambda(H) + \varepsilon$. Hence by definition of regular θ-content, λ is a regular θ-content. So there exists a unique regular θ-Borel measure μ such that $\mu(H) = \lambda(H)$ for every H-set H.

Next we show that if $f \in \mathcal{H}$ and $f \geq 0$, then $\int f \, d\mu \leq F(f)$ ...... (1). To prove this inequality, by linearity it is sufficient to prove for functions f such that $0 \leq f \leq 1$.

For a fixed positive integer m, and for any x for which $f(x) \in \left[\frac{i-1}{m}, \frac{i}{m}\right]$ we define

$$f_k(x) = \begin{cases} 1 \text{ if } 1 \leq k \leq i-1 \\ 0 \text{ if } i+1 \leq k \leq m \end{cases}$$

and $$f_i(x) = mf - (i-1).$$

Clearly, $f(x) = \frac{1}{m}\sum_{k=1}^{m} f_k(x)$, for every $x \in X$. Since f is continuous and hence θ-continuous, so the sets $U_k = f^{-1}\left(\frac{k}{m},\infty\right)$ are θ-open, for $k = 0, 1, 2, \ldots, m$ and also U_k's are monotone decreasing subsets of X with $U_m = \phi$ (as $0 \leq f \leq 1$).

We claim that $\mu(U_k) \leq F(f_k)$. If H be an H-set contained in U_k, then $F_k \geq H$ and so

$$\mu(H) = \lambda(H) \leq F(f_k);$$

and hence

$$\mu(U_k) = \sup \mu(H) \leq F(f_k).$$

Therefore we can write

$$
\begin{aligned}
F(f) &= F\left(\frac{1}{m}\sum_{k=1}^{m} f_k\right) \\
&= \frac{1}{m}\sum_{k=1}^{m} F(f_k) \\
&\geq \frac{1}{m}\sum_{k=1}^{m} \mu(U_k) \\
&= \sum_{k=1}^{m}\left(\frac{k}{m} - \frac{k-1}{m}\right)\mu(U_k) \\
&= \sum_{k=1}^{m-1}\frac{k}{m}\left[\mu(U_k) - \mu(U_{k+1})\right] \\
&= \sum_{k=1}^{m-1}\frac{k+1}{m}\mu\left(U_k - U_{k+1}\right) - \frac{1}{m}\mu(U_1) \\
&\geq \sum_{k=1}^{m-1}\int_{U_k - U_{k-1}} f\,d\mu - \tfrac{1}{m}\mu(U_1)
\end{aligned}
$$

$$= \int_{U_1} f\,d\mu - \tfrac{1}{m}\mu(U_1)$$

$$\geq \int f\,d\mu - \tfrac{1}{m}\mu(U_0)$$

Since $\mu(U_0)$ is finite and m is arbitrary, we have $\int f\,d\mu \leq F(f)$.

To complete our proof, it remains to show that $F(f) \leq \int f\,d\mu$, for every $f \in \mathcal{H}$, because the reverse inequality i.e., $\int f\,d\mu \leq F(f)$ follows by applying that inequality to $-f$.

So let f be a function in $\mathcal{H}$ and H be its H-set support. The definition of λ shows the existence of a function $h \in \mathcal{H}$ such that $h \geq H$ and $F(h) \leq \lambda(H) + \varepsilon$. If $h_0 = \min\{h,1\}$, then

$$F(h_0) \leq F(h) \leq \lambda(H) + \varepsilon \leq \int h_0\,d\mu + \varepsilon$$

Since $h_0 \geq H$, then $fh_0 = f$. Since f is real valued continuous function and hence θ-continuous so f carries H-set to H-sets and hence is closed and bounded there, $|f(x)| \leq \beta$, for all $x \in X$, then $h_0\,(f + \beta) \geq 0$ and $h_0(f + \beta) \in \mathcal{H}$. So by (1),

$$\begin{aligned} \int (f+\beta)h_0 d\mu &\leq F\big((f+\beta)h_0\big) \\ &= F(fh_0) + F(\beta h_0) \\ &= F(f) + \beta F(h_0) \end{aligned}$$

Therefore,

$$\begin{aligned} \int (f+\beta)h_0 d\mu &= \int f\,d\mu + \beta \int h_0\,d\mu \\ &\leq F(f) + \beta F(h_0) \end{aligned}$$

i.e.,

$$\begin{aligned} F(f) &\geq \int f\,d\mu + \beta\Big[\int h_0 d\mu - F(h_0)\Big] \\ &\geq \int f\,d\mu - \beta\varepsilon \;\; [\text{since, } F(h_0) \leq \int h_0\,d\mu + \varepsilon] \end{aligned}$$

Since ε is arbitrary,

$$\int f\, d\mu \leq F(f)$$

Hence the theorem is complete.

ACKNOWLEDGEMENT

The author expresses his earnest gratitude to Prof. S. Ganguly, Department of Pure Mathematics, University of Calcutta, for his guidance throughout the preparation of this paper and to Dr. C. K. Basu, Department of Mathematics, Kalyani University, for his valuable suggestions.

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Jour. Pure Math., Vol. 19, 2002, pp. 61-71

ON RELATIVE DEFECTS OF MEROMORPHIC FUNCTIONS

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ABSTRACT : In the paper we consider two different meromorphic functions having common roots and find some relations involving the relative defects. We also study the relationship between the usual defect and the relative defect of a meromorphic function corresponding to distinct zeros and distinct poles with the Nevanlinna defect.

Mathematics Subject Classification (2000) : 30D35, 30D30.

Key words and phrases : Meromorphic function, Relative defect, Nevanlinna defect.

1. INTRODUCTION AND NOTATIONS

Let f_1 and f_2 be two non-constant meromorphic functions defined in the open complex plane C. Let $n_0(r,a)$ denoted the number of common roots in the disk $|z| \le r$ of the two equations $f_1 = a$ and $f_2 = a$ where a is any complex number and $\bar{n}_0(r,a)$ denotes the number of distinct common roots in the disk $|z| \le r$ of the two equations $f_1 = a$ and $f_2 = a$. A. P. Singh [3] found some relations on relative defects corresponding to the common roots of two meromorphic functions. In the paper we further investigate the results of Singh [3] and prove some new results on relative defects of the common roots of $f_1 = a$ and $f_2 = a$.

To start our paper we require the following :

$$\text{Let } \overline{N}_0(r,a) = \int_0^r \frac{\bar{n}_0(t,a) - \bar{n}_0(0,a)}{t} dt + \bar{n}_0(0,a)\log r$$

$$\overline{N}_{1,2}(r,a) = \overline{N}(r, \frac{1}{f_1 - a}) + \overline{N}(r, \frac{1}{f_2 - a}) - 2\overline{N}_0(r,a).$$

Also let $\bar{n}_0^{(k)}(r,a)$, $\overline{N}_{1,2}^{(k)}(r,a)$ etc. denote the corresponding quantities with respect to $f_1^{(k)}$ and $f_2^{(k)}$, where k is any non negative integer.

Now we set the following quantities :

$$\delta_{1,2}(a) \quad = \quad 1 - \limsup_{r \to \infty} \frac{N_{1,2}(r,a)}{T(r,f_1) + T(r,f_2)}$$

$$\delta_{1,2}^{(k)}(a) = 1-\lim_{r\to\infty}\sup\frac{N_{1,2}^{(k)}(r,a)}{T(r,f_1)+T(r,f_2)}$$

$$\delta_0(a) = 1-\lim_{r\to\infty}\sup\frac{N_0(r,a)}{T(r,f_1)+T(r,f_2)}$$

$$\Theta_{1,2}(a) = 1-\lim_{r\to\infty}\sup\frac{\overline{N}_{1,2}(r,a)}{T(r,f_1)+T(r,f_2)}$$

$$\Theta_{1,2}^{(k)}(a) = 1-\lim_{r\to\infty}\sup\frac{\overline{N}_{1,2}^{(k)}(r,a)}{T(r,f_1)+T(r,f_2)}$$

$$\Theta_0(a) = 1-\lim_{r\to\infty}\sup\frac{\overline{N}_0(r,a)}{T(r,f_1)+T(r,f_2)}$$

Let $\alpha \in C\cup\{\infty\}$ For a non-constant meromorphic function f, the Nevanlinna defect $\delta(\alpha, f)$ of α is defined in the following manner :

$$\delta(\alpha; f) = 1-\lim_{r\to\infty}\sup\frac{N(r,\alpha;f)}{T(r,f)} = \lim_{r\to\infty}\inf\frac{m(r,\alpha;f)}{T(r,f)}$$

The term $1-\lim_{r\to\infty}\sup\frac{\overline{N}(r,\alpha;f)}{T(r,f)}$ is called the usual defect of α corresponding to distinct zeros and is denoted by $\Theta(\alpha, f)$. The usual defect of α corresponding to distinct poles is similarly defined. A. P. Singh [2] introduced the term relative defect for distinct zeros and poles and established various relations between it, relative defects and the usual defects. In the paper we establish some results on the relationship between Nevanlinna defect, usual and relative defect corresponding to distinct zeros and poles in the line of Singh [2].

The following definition is well known.

Definition 1.1. The relative defect of 'α' with respect to $f^{(k)}$ corresponding to simple zeros is defined as

$$\Theta_r^{(k)}(\alpha;f) = 1-\lim_{r\to\infty}\sup\frac{\overline{N}(r,\alpha;f^{(k)})}{T(r,f)} \text{ for } k = 1, 2, 3, \ldots$$

We do not explain the definitions and standard notations of Nevanlinna theory because those are available in [1]. The term S(r,f) denotes any quantity satisfying $S(r,f) = o\{T(r,f)\}$

as $r \to \infty$ through all values of r if f is of finite order and except possibly for a set of r of finite linear measure otherwise.

2. LEMMA

In this section we present a lemma which will be needed in the sequel.

The following lemma is due to Millox {p.55, [1]}.

Lemma 2.1.[1]. Let k be a positive integer and $\psi = \sum_{i=0}^{k} a_i f^{(i)}$ where a_i are meromorphic functions such that $T(r,a_i) = S(r,f)$. Then $m\left(r, \frac{\psi}{f}\right) = S(r,f)$

3. THEOREMS

In this section we present the main results of the paper.

Theorem 3.1. Let f_1 and f_2 be any two meromorphic functions such that $\overline{N}(r,f_1) = S(r,f_1)$ and $\overline{N}(r,f_2) = S(r,f_2)$. Also let a_i (i = 1, 2, ...p) and b_j (j = 1, 2, ... q) be two sets of distinct finite non zero complex numbers. Then for any positive integer k.

$$\Theta_{1,2}(0) + q\delta_{1,2}(0) + \sum_{i=1}^{p} \Theta_{1,2}(a_i) + \sum_{j=1}^{q} \Theta_{1,2}{}^{(k)}(b_j) +$$

$$2\left\{\Theta_0(a) + q\delta_0(a) + \sum_{i=1}^{p} \Theta_0(a_i) + \sum_{i=1}^{q} \Theta_0^{(k)}(b_j)\right\} \le 3 + 2p + 5q.$$

Proof. From Nevanlinna's first fundamental theorem and by Lemma 2.1 we get,

$$T(r,f) = T(r,\frac{1}{f}) + O(1) \le N(r,\frac{1}{f}) + m(r,\frac{f^{(k)}}{f}) + m(r,\frac{1}{f^{(k)}}) + O(1)$$

Thus $T(r,f) \le N(r,\frac{1}{f}) + T(r,\frac{1}{f^{(k)}}) - N(r,\frac{1}{f^{(k)}}) + S(r,f)$

i.e., $T(r,f) \le N(r,\frac{1}{f}) + T(r,f^{(k)}) - N(r,\frac{1}{f^{(k)}}) + S(r,f)$ ... (1)

Again by Nevanlinna's second fundamental theorem,

$$qT(r,f^{(k)}) \le \overline{N}(r,f^{(k)}) + \sum_{j=1}^{q} \overline{N}(r,\frac{1}{f^{(k)}-b_j}) + \overline{N}(r,\frac{1}{f^{(k)}}) + S(r,f) \quad \ldots (2)$$

Therefore from (1) and (2) we obtain,

$$qT(r,f) \le \overline{N}(r,f^{(k)}) + \overline{N}(r,\frac{1}{f^{(k)}}) - qN(r,\frac{1}{f^{(k)}}) + \sum_{j=1}^{q} \overline{N}(r,\frac{1}{f^{(k)}-b_j})$$

$$+qN(r;\frac{1}{f}) + S(r,f) \quad \ldots (3)$$

Since $\overline{N}(r,f^{(k)}) = \overline{N}(r,f)$ and $\overline{N}(r,\frac{1}{f^{(k)}}) - qN(r,\frac{1}{f^{(k)}}) \le 0$ it follows from (3),

$$qT(r,f) \le \sum_{j=1}^{q} \overline{N}(r,\frac{1}{f^{(k)}-b_j}) + \overline{N}(r,f) + qN(r,\frac{1}{f}) + S(r,f) \quad \ldots (4)$$

In view of Nevanlinna's second fundamental theorem,

$$pT(r,f) \le \overline{N}(r,f) + \overline{N}(r,\frac{1}{f}) + \sum_{i=1}^{p} \overline{N}(r,\frac{1}{f-a_i}) + S(r,f) \quad \ldots (5)$$

Adding (4) and (5) it follows that,

$$(p+q)T(r,f) \le \overline{N}(r,\frac{1}{f}) + qN(r,\frac{1}{f}) + 2\overline{N}(r,f) + \sum_{i=1}^{p} \overline{N}(r,\frac{1}{f-a_i})$$

$$+\sum_{j=1}^{q} \overline{N}(r,\frac{1}{f^{(k)}-b_j}) + S(r,f) \quad \ldots (6)$$

Applying this inequality for f_1 and f_2 we get,

$(p + q)\{T(r,f_1) + T(r,f_l)\}$

$$\le \left\{\overline{N}(r,\frac{1}{f_1}) + \overline{N}(r,\frac{1}{f_2})\right\} + q\left\{N(r,\frac{1}{f_1}) + N(r,\frac{1}{f_2})\right\} + 2\{\overline{N}(r,f_1) + \overline{N}(r,f_2)\}$$

$$\sum_{i=1}^{p}\left\{\overline{N}(r,\frac{1}{f_1-a_i}) + \overline{N}(r,\frac{1}{f_2-a_i})\right\} + \sum_{j=1}^{q}\left\{\overline{N}(r,\frac{1}{f_1^{(k)}-b_j}) + \overline{N}(r,\frac{1}{f_2^{(k)}-b_j})\right\}$$

$+ S(r,f_1) + S(r,f_2)$

$$+\sum_{i=1}^{p}\{\overline{N}_{1,2}(r,0)+2\overline{N}_0(r,a_i)\}+\sum_{j=1}^{q}\{(N_{1,2}^{(k)}(r,b_j)+2\overline{N}_0(r,b_j)\}+S(r,f_1)+S(r,f_2)$$

$$+\sum_{i=1}^{p}\{\overline{N}_{1,2}(r,a_i)+2\overline{N}_0(r,a_i)\}+\sum_{j=1}^{q}\{N_{1,2}^{(k)}(r,b_j)+2\overline{N}_0(r,b_j)\}+S(r,f_1)+S(r,f_2) \quad \ldots (7)$$

Now dividing both sides of (7) by $T(r,f_1) + T(r,f_2)$ and taking limit superior we get,

$$(p+q)\leq\{1-\Theta_{1,2}(0)\}+2\{1-\Theta_0(a)\}+q\{1-\delta_{1,2}(0)\}+2q\{1-\delta_0(a)\}$$

$$+\sum_{i=1}^{p}\{1-\Theta_{1,2}(a_i)\}+2\sum_{i=1}^{p}\{1-\Theta_0(a_i)\}+\sum_{j=1}^{q}\{1-\Theta_{1,2}^{(k)}(b_j)\}+2\sum_{j=1}^{q}\{1-\Theta_0^{(k)}(b_j)\}$$

i.e., $p+q\leq 3-\Theta_{1,2}(0)-2\Theta_0(a)+3q-q\delta_{1,2}(0)-2q\delta_0(a)+3p-$

$$\sum_{i=1}^{p}\Theta_{1,2}(a_i)\}-2\sum_{i=1}^{p}\Theta_0(a_i)+3q-\sum_{j=1}^{q}\Theta_{1,2}^{(k)}(b_j)\}-2\sum_{j=1}^{q}\Theta_0^{(k)}(b_j)$$

i.e., $$\Theta_{1,2}(0)+q\delta_{1,2}(0)+\sum_{i=1}^{p}\Theta_{1,2}(a_i)+\sum_{j=1}^{q}\Theta_{1,2}^{(k)}(b_j)+$$

$$2\{\Theta_0(a)+q\delta_0(a)+\sum_{i=1}^{p}\Theta_0(a_i)+\sum_{j=1}^{q}\Theta_0^{(k)}(b_j)\}\leq 3+2p+5q$$

This proves the theorem.

Theorem 3.2. For any two meromorphic function f_1 and f_2 with $\overline{N}(r,f_1) = S(r,f_1)$ and $\overline{N}(r,f_2) = S(r,f_2)$

$$\Theta_{1,2}^{(k)}(0)+\sum_{j=1}^{q}\Theta_{1,2}^{(k)}(b_j)+q\sum_{i=1}^{p}\delta_{1,2}(a_i)+2\{\Theta_0^{(k)}(0)+\sum_{j=1}^{q}\Theta_0^{(k)}(b_j)+q\sum_{i=1}^{p}\delta_0(a_i)\}$$

$$\leq 3+3q+2pq$$

where a_i ($i = 1, 2, \ldots\ldots p$) and b_j ($j = 1, 2, \ldots q$) be any two sets of distinct finite non zero complex numbers and k is any positive integer.

Proof. Let us consider $F=\sum_{i=1}^{p}\frac{1}{f-a_i}$

Then by Lemma 1 we obtain,

$$\sum_{i=1}^{p} m\left(r, \frac{1}{f-a_i}\right) \leq m(r, F) + O(1)$$

i.e.,
$$\sum_{i=1}^{p} m\left(r, \frac{1}{f-a_i}\right) \leq m(r, Ff^{(k)}) + m\left(r, \frac{1}{f^{(k)}}\right) + O(1)$$

i.e.,
$$\sum_{i=1}^{p} m\left(r, \frac{1}{f-a_i}\right) \leq \sum_{i=1}^{p} m(r, \frac{f^{(k)}}{f-a_i}) + m\left(r, \frac{1}{f^{(k)}}\right) + O(1)$$

i.e.,
$$\sum_{i=1}^{p} m\left(r, \frac{1}{f-a_i}\right) \leq m\left(r, \frac{1}{f^{(k)}}\right) + S(r, f) \qquad \ldots (8)$$

Adding to $\sum_{i=1}^{p} N\left(r, \frac{1}{f-a_i}\right)$ to both sides of (8) we get,

$$\sum_{i=1}^{p} T\left(r, \frac{1}{f-a_i}\right) \leq T\left(r, \frac{1}{f^{(k)}}\right) + \sum_{i=1}^{p} N\left(r, \frac{1}{f-a_i}\right) + S(r, f)$$

i.e.,
$$pqT(r,f) \leq qT(r,f^{(k)}) + q\sum_{i=1}^{p} N(r, \frac{1}{f-a_i}) + S(r,f) \qquad \ldots (9)$$

Again by Nevanlinna's second fundamental theorem and in view of $\overline{N}(r, f^{(k)}) = \overline{N}(r, f)$ we obtain,

$$qT(r, f^{(k)}) \leq \overline{N}(r, f) + \overline{N}(r, \frac{1}{f^{(k)}}) + \sum_{j=1}^{q} \overline{N}\left(r, \frac{1}{f-b_j}\right) + S(r, f) \qquad \ldots (10)$$

Now from (9) and (10) it follows that,

$$pq\, T(r,f) \leq \overline{N}(r,f) + \overline{N}\left(r, \frac{1}{f^{(k)}}\right) + q\sum_{i=1}^{p} N\left(r, \frac{1}{f-a_j}\right) +$$

$$\sum_{j=1}^{q} \overline{N}\left(r, \frac{1}{f^{(k)}-b_j}\right) + S(r, f) \qquad \ldots (11)$$

Applying (11) for f_1 and f_2 we get in view of $\overline{N}(r,f_1) = S(r,f_1)$ and $\overline{N}(r,f_2) = S(r,f_2)$ $pq\{T(r,f_1) + T(r,f_2)\}$

$$\le \left\{\overline{N}\left(r,\frac{1}{f_1^{(k)}}\right)+\overline{N}\left(r,\frac{1}{f_2^{(k)}}\right)\right\}+\sum_{j=1}^{q}\left\{\overline{N}\left(r,\frac{1}{f_1^{(k)}-b_j}\right)+\overline{N}\left(r,\frac{1}{f_2^{(k)}-b_j}\right)\right\}$$

$$+\ q\sum_{i=1}^{p}\left\{N\left(r,\frac{1}{f_1-a_i}\right)+\overline{N}\left(r,\frac{1}{f_2-a_i}\right)\right\}\ +\ S(r,f_1)\ +\ S(r,f_2)$$

$$= \{\overline{N}_{1,2}^{(k)}(r,0)+2\overline{N}_0^{(k)}(r,0)\}+\{\overline{N}_{1,2}^{(k)}(r,b_j)+2\overline{N}_0^{(k)}(r,b_j)\}$$

$$+q\sum_{i=1}^{p}\{N_{1,2}(r,a_i)+2N_0(r,a_i)\}+S(r,f_1)+S(r,f_2) \qquad \dots (12)$$

Dividing both sides of (12) by $T(r,f_1) + T(r,f_2)$ and taking limit superior we obtain,

$$pq\le\{1-\Theta_{1,2}^{(k)}(0)\}+2\{1-\Theta_0^{(k)}(0)\}+\sum_{j=1}^{q}\{1-\Theta_{1,2}^{(k)}(b_j)\}$$

$$+\ 2\sum_{j=1}^{q}\{1-\Theta_0^{(k)}(b_j)\}+q\sum_{i=1}^{p}\{1-\delta_{1,2}(a_i)\}+2q\sum_{i=1}^{p}\{1-\delta_0(a_i)\}.$$

i.e., $$pq\le 3-\Theta_{1,2}^{(k)}(0)+2\Theta_0^{(k)}(0)+3q-\sum_{j=1}^{q}\Theta_{1,2}^{(k)}(b_j)-\sum_{j=1}^{q}\Theta_0^{(k)}(b_j)$$

$$+3pq-q\sum_{i=1}^{p}\delta_{1,2}(a_i)-2q\sum_{i=1}^{p}\delta_0(a_i)$$

i.e., $$\Theta_{1,2}^{(k)}(0)+\sum_{j=1}^{q}\Theta_{1,2}^{(k)}(b_j)+q\sum_{i=1}^{p}\delta_{1,2}(a_i)+$$

$$2\left\{\Theta_0^{(k)}(0)+\sum_{j=1}^{q}\Theta_0^{(k)}(b_j)+q\sum_{i=1}^{p}\delta_0(a_i)\right\}\le 3\ +\ 3q\ +\ 2pq.$$

Thus the theorem is proved.

Theorem 3.3. Let a_i (i = 1, 2, ...p) and b_j = (1, 2, ... q) be any two sets of finite non zero distinct complex numbers. Then for any two meromorphic functions f_1 and f_2 with

$\overline{N}(r,f_1) = S(r,f_1)$, $\overline{N}(r,f_2) = S(r,f_2)$

$$\Theta_{1,2}^{(k)}(0)+2\Theta_0^{(k)}(0)+\Theta_{1,2}(0)+2\Theta_0(r,0)+q\delta_{1,2}(0)+2q\delta_0(0)+(1+q)\sum_{i=1}^{p}\delta(a_i)$$

$$+\ 2(1+q)\sum_{i=1}^{p}\Theta_0(a_i)+2\sum_{j=1}^{q}\Theta_{1,2}^{(k)}(b_j)+4\sum_{j=1}^{q}\Theta_0^{(k)}(b_j)\le 6 + 2p + 8q + 2pq$$

where k is any positive integer.

Proof. Adding (6) and (11) we get,

(p + q + pq) T(r,f)

$$\le 3\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f^{(k)}}\right)+\overline{N}\left(r,\frac{1}{f}\right)+qN\left(r,\frac{1}{f}\right)$$

$$+(1+q)\sum_{i=1}^{p}N\left(r,\frac{1}{f-a_i}\right)+2\sum_{j=1}^{q}\overline{N}\left(r,\frac{1}{f^{(k)}-b_j}\right)+S(r,f) \quad \ldots (13)$$

Applying this inequality for f_1 and f_2 it follows that,

(p + q + pq){T(r,f_1) + T(r,f_2)}

$$\le\left\{\overline{N}\left(r,\frac{1}{f_1^{(k)}}\right)+\overline{N}\left(r,\frac{1}{f_2^{(k)}}\right)\right\}+\left\{\overline{N}\left(r,\frac{1}{f_1}\right)+\overline{N}\left(r,\frac{1}{f_2}\right)\right\}$$

$$+q\left\{N\left(r,\frac{1}{f_1}\right)+N\left(r,\frac{1}{f_2}\right)\right\}+(1+q)\sum_{i=1}^{p}\left\{N\left(r,\frac{1}{f_1-a_i}\right)+N\left(r,\frac{1}{f_2-a_i}\right)\right\}$$

$$+2\sum_{i=1}^{q}\left\{N\left(r,\frac{1}{f_1^{(k)}-b_j}\right)+\overline{N}\left(r,\frac{1}{f_1^{(k)}-b_j}\right)\right\} + S(r,f_1) + S(r,f_2)$$

$$=\{\overline{N}_{1,2}^{(k)}(r,0)+2\overline{N}_0^{(k)}(r,0)\}+\{\overline{N}_{1,2}(r,0)+2\overline{N}_0(r,0)\}+$$

$$q\{N_{1,2}(r,0)+2N_0(r,0)\}+(1+q)\sum_{i=1}^{p}\{N_{1,2}(r,a_i)+2\overline{N}_0(r,a_i)\}$$

$$+2\sum_{j=1}^{q}\{\overline{N}_{1,2}^{(k)}(r,b_j)+2\overline{N}_0^{(k)}(r,b_j)\} + S(r,f_1) + S(r,f_2) \qquad \ldots(14)$$

On dividing both sides of (14) by $T(r,f_1) + T(r,f_2)$ and taking limit superior we get,

$$p + q + pq \leq \{1-\Theta_{1,2}^{(k)}(0)\}+2\{1-\Theta_0^{(k)}(0)\}+\{1-\Theta_{1,2}^{(k)}(0)\}+2\{1-\Theta_0^{(k)}(r,0)\}$$

$$+q\{1-\delta_{1,2}(0)\}+2q\{1-\delta_0(0)\}+(1+q)\left\{p-\sum_{i=1}^{p}\delta(a_i)\right\}$$

$$+2(1+q)\left\{p-\sum_{i=1}^{p}\Theta_0(a_i)\right\}+2\left\{q-\sum_{j=1}^{q}\Theta_{1,2}^{(k)}(b_j)\right\}+\left\{4q-\sum_{j=1}^{q}\Theta_0^{(k)}(b_j)\right\}$$

i.e.,
$$\Theta_{1,2}^{(k)}(0)+2\Theta_0^{(k)}(0)+\Theta_{1,2}(0)+2\Theta_0(r,0)+q\delta_{1,2}(0)+2q\delta_0(0)$$

$$+(1+q)\sum_{i=1}^{p}\delta(a_i)+2(1+q)\sum_{i=1}^{p}\Theta_0(a_i)+2\sum_{j=1}^{q}\Theta_{1,2}^{(k)}(b_j)+4\sum_{j=1}^{q}\Theta_0^{(k)}(b_j)$$

$$\leq 6+2p+8q+2pq.$$

This proves the theorem.

The next theorem is based on the relationship between the usual defect and the relative defect of a meromorphic function corresponding to distinct zeros and distinct poles with the Nevanlinna defect.

Theorem 3.4. Let f ba a meromorphic function and a_i (i = 1, 2, ...p), b_j (j = 1, 2, ... q) are finite distinct complex numbers such that $a_i \neq 0$, $b_j \neq 0$. Then for any positive integer k,

$$3\,\Theta(\infty,f)+\Theta_r^{(k)}(0,f)+\Theta(0,f)\,q\,\delta(0,f)+$$

$$(1+q)\sum_{i=1}^{p}\delta(a_i,f)+2\sum_{j=1}^{q}\Theta_r^{(k)}(b_j,f)\leq 2q+5.$$

Proof. From (13) we obtain

$$p + q + pq \leq 3\ \limsup_{r\to\infty}\frac{\overline{N}(r,f)}{T(r,f)}+\limsup_{r\to\infty}\frac{\overline{N}\left(r,\frac{1}{f^{(k)}}\right)}{T(r,f)}$$

$$+\limsup_{r\to\infty}\frac{\overline{N}\left(r,\frac{1}{f}\right)}{T(r,f)}+q\limsup_{r\to\infty}\frac{N\left(r,\frac{1}{f}\right)}{T(r,f)}$$

$$+(1+q)\sum_{i=1}^{p}\limsup_{r\to\infty}\frac{N\left(r,\frac{1}{f-a_i}\right)}{T(r,f)}$$

$$+2\sum_{j=1}^{q}\limsup_{r\to\infty}\frac{\overline{N}\left(r,\frac{1}{f^{(k)}-b_j}\right)}{T(r,f)}$$

i.e., p + q + pq

$$\leq 3\{1-\Theta(\infty,f)\}+\{1-\Theta_r^{(k)}(0,f)\}+\{1-\Theta(0,f)\}+q\{1-\delta(0,f)\}+$$

$$(1+q)\{p-\sum_{i=1}^{p}\delta(a_i,f)\}+2\{q-\sum_{j=1}^{q}\Theta_r^{(k)}(b_j,f)\}$$

i.e., $3\Theta(\infty,f)+\Theta_r^{(k)}(0,f)+\Theta(0,f)+q\delta(0,f)+$

$$(1+q)\sum_{i=1}^{p}\delta(a_i,f)+2\sum_{j=1}^{q}\Theta_r^{(k)}(b_j,f)\leq 2q+5 \qquad \ldots (15)$$

This proves the theorem.

The following corollary is a consequence of Theorem 3.4.

Corollary 3.5. If f is an entire function such that $\sum_{i=1}^{p}\delta(a_i,f)=1$, then for any positive integer k,

$$\Theta_r^{(k)}(0,f)+\Theta(0,f)+q\,\delta(0,f)+2\sum_{j=1}^{q}\Theta_r^{(k)}(b_j,f)\leq q+1.$$

where a_i and b_j have the same meaning as in Theorem 3.4.

Proof. Since f is entire function $\Theta(\infty,f)=1$.

Now in view of $\sum_{i=1}^{p} \delta(a_i, f) = 1$ and $\Theta(\infty, f) = 1$ it follows from (15),

$$3 + \Theta_r^{(k)}(0, f) + \Theta(0, f) + q\ \delta(0, f) + (1 + q)$$

$$+2 \sum_{j=1}^{q} \Theta_r^{(k)}(b_j, f) \leq 2q + 5$$

from which the corollary follows.

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Jour. Pure Math., Vol. 19, 2002, pp. 73-79

PRODUCT PSEUDO ALGEBRAIC SPACES

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ABSTRACT : If (X,T_1) and (Y,T_2) be two p-topological spaces, then $T^* = \{A_1 \times A_2 : A_1 \in T_1, A_2 \in T_2\}$ is defined to be a Product p-topology on $X \times Y$. Finally Product p-topological space and Product p-a space are established and some of its properties are discussed here.

Key words and phrases : Pseudo topological space, Product pseudo topological space and Product pseudo algebraic space are denoted by p-topological space, Product p-topological space and Product p-a space.

1. INTRODUCTION

The aim of this paper is to introduce the notion of Pseudo algebraic spaces. A pseudo algebraic space is defined to be a non-empty set having two types of structures—a Pseudo topological structure and a Pseudo algebraic structure. We have introduced the notion of sub p-a spaces, keeping it in mind that notions of topological subspaces and subgroups go together. We have introduced a special kind of mapping from one p-a space to another. These mappings are counter-part of homomorphism in algebra and continuous functions in topology. We call these mappings p-a homomorphisms. One of the purposes of the topic is to study the elementary properties of these p-a homomorphisms and form a basis for further study of p-a spaces.

Our special interest would be on Product p-topological space. We have introduced a p-a structure on Product p-topological space. A Product p-topological space equipped with a p-a structure is called a Product p-a space.

2. PRELIMINARIES

Definition 2.1. Let X be a non-empty set and T a class of subsets of X such that

(i) $X \in T$

(ii) there exists an $A_0 \in T$ such that $A_0 \subseteq A$ for every $A \in T$

(iii) any finite intersection of members of T is a member of T.

The class T is called a Pseudo topology (p-topology) and the pair (X,T) is called a p-topological space. When there is no scope for confusion, X may be simply called a

p-topological space. The members of T are called Pseudo open set (p-open sets) of X. A set A_0 with the property (ii) is called a minimal p-open set. In a p-topological space, there is one and only one minimal p-open set. Therefore, a minimal p-open set is referred as the minimal p-open set.

Definition 2.2. A p-topological space (X,T) is said to have a Pseudo algebraic structure (p-a structure) if there exists a Pseudo algebraic function,

$\alpha : P^X \times P^X \to P^X$ (P^X is the power set of X)

satisfying the following conditions.

(i) $\alpha(\alpha(A,B), C) = \alpha(A, \alpha(B,C))$, $A,B,C, \in P^X$

(ii) $\alpha(A,B) \in T$ if $\alpha(A,B) = \alpha(B,A)$ for $A, B \in T$

(iii) if $A_1 \subseteq A$, $B_1 \subseteq B$ then $\alpha(A_1, B_1) \subseteq \alpha(A,B)$

(iv) $\alpha(A_0,A) = \alpha(A,A_0)$, $A \in P^X$, where A_0 is the minimal p-open set.

We say that the triple (X,T, α) is a p-topological space with a p-a structure α or simply a Pseudo algebraic space (p-a space).

Definition 2.3. A subset A in a p-a space (X,T, α) is called a normal set if $\alpha(A,Y) = \alpha(Y,A)$, $\forall\ Y \in P^X$.

Definition 2.4. The p-topology T of a p-a space (X,T, α) is said to be normal if every p-open set is normal and a p-a space is said to be normal if its p-topology is normal.

Definition 2.5. The p-topology T′ on Y which is a non-empty subset of X defined by

$T' = \{ A \cap Y : A \in T\}$ is called the relative p-topology on Y and the p-topological space (Y, T′) is called a sub p-topological space of (X,T).

Definition 2.6. A sub p-topological space (Y,T′) is called a sub p-a space of a p-a space (X,T, α) with a p-a structure α′ if α induces a p-a function α′ on P^Y such that

(i) $\alpha'(A,B) = \alpha(A,B)$, $A,B \in P^Y$

(ii) $\alpha'(A,B) \in T'$ if $\alpha'(A,B) = \alpha'(B,A)$ for $A,B \in T'$

and (iii) $\alpha'(A'_0,A) = \alpha(A,A'_0)$, $A \in P^Y$, where A'_0 is the minimal p-open set in T′.

Definition 2.7. Let (X,T) and (Y,T*) be two p-topological spaces. Let f be a maping from X to Y. We say that the mapping f is p-continuous if $f^{-1}(A^*) \in T$ whenever $A^* \in T^*$ and is called p-open if $f(A) \in T^*$ whenever $A \in T$.

Definition 2.8. Let (X,T,α) and (Y,T*,β) be two p-a spaces.

A function $f : X \to Y$ is called a p-a homomorphism if it is such that

(i) f is both p-open and p-continuous

(ii) $f(\alpha(A,B)) = \beta(f(A),f(B))$, $A, B \in P^X$

and (iii) $\alpha(f^{-1}(A^*), f^{-1}(B^*)) = f^{-1}(\beta(A^*,B^*))$, $A^*, B^* \in P^Y$.

3. PRODUCT PSEUDO ALGEBRAIC SPACES

Now we establish the Product p-topological space and Product p-a space.

Proposition 3.1. Let (X,T_1) and (Y,T_2) be two p-topological spaces. Let $T^* = \{A_1 \times A_2 : A_1 \in T_1, A_2 \in T_2\}$. Then T^* is a p-topology on $X \times Y$.

Proof.

(i) $X \in T_1, Y \in T_2 \Rightarrow X \times Y \in T^*$

(ii) Let A_0 and B_0 be the minimal p-open sets of T_1 and T_2 respectively. Then $A_0 \times B_0 \in T^*$. is the minimal p-open set of T^*.

(iii) Let $A_1 \times A_2, B_1 \times B_2 \in T^*$

$(A_1 \times A_2) \cap (B_1 \times B_2) = (A_1 \cap B_1) \times (A_2 \cap B_2) \in T^*$

since $A_1 \in T_1, B_1 \in T_1 \Rightarrow A_1 \cap B_1 \in T_1$

and $A_2 \in T_2, B_2 \in T_2 \Rightarrow A_2 \cap B_2 \in T_2$

T^* is a p-topology on $X \times Y$.

Remark 3.2. T^* is called the Product p-topology on $X \times Y$ and $(X \times Y, T^*)$ is called the Product p-topological space.

Proposition 3.3. Let (X,T_1,α_1) and (Y,T_2,α_1) be two p-a spaces where α_1, and α_2 are p-a structure on X and Y respectively.

Let $T^* = \{ A_1 \times A_2 : A_1 \in T_1, A_2 \in T_2\}$ be a p-topology on $X \times Y$.

Let $\alpha^* : P^{X\times Y} \times P^{X\times Y} \to P^{X\times Y}$ be such that

$\alpha^*(A_1 \times A_2, B_1 \times B_2) = \alpha_1(A_1,B_1) \times \alpha_2(A_2,B_2)$

Then $(X \times Y, T^*, \alpha^*)$ is a p-a space.

Proof. We show that α^* is a p-a structure on $X \times Y$

(i) $\alpha^*(\alpha^*(A_1 \times A_2, B_1 \times B_2), C_1 \times C_2)$

$= \alpha^*(\alpha_1(A_1,B_1) \times \alpha_2(A_2,B_2), C_1 \times C_2)$

$= \alpha_1(\alpha_1((A_1,B_1), C_1) \times \alpha_2(\alpha_2((A_2,B_2),C_2))$

$= \alpha_1(A_1,\alpha_1(B_1,C_1)) \times \alpha_2(A_2,\alpha_2(B_2,C_2))$

$= \alpha^*(A_1 \times A_2, \alpha_1(B_1,C_1) \times \alpha_2(B_2,C_2))$

$= \alpha^*(A_1 \times A_2, \alpha^*(B_1 \times B_2, (C_1 \times C_2))$

(ii) $\alpha^*(A_1 \times A_2, B_1 \times B_2) = \alpha_1(A_1,B_1) \times \alpha_2(A_2.B_2)$

$= \alpha_1(B_1,A_1) \times \alpha_2(B_2,A_2)$

$= \alpha^*(B_1 \times B_2, A_1 \times A_2)$

$\therefore\ \alpha^*(A_1 \times A_2, B_1 \times B_2) \in T^*$

(iii) Let $A_1 \subseteq C_1$, $A_2 \subseteq C_2$ and $B_1 \subseteq D_1$, $B_2 \subseteq D_2$ then

$\alpha^*(A_1 \times A_2, B_1 \times B_2) = \alpha_1(A_1,B_1) \times \alpha_2(A_2,B_2)$

$\subseteq \alpha_1(C_1,D_1) \times \alpha_2(C_2,D_2)$

$= \alpha^*(C_1 \times C_2, D_1 \times D_2)$

(iv) $\alpha^*(A_0 \times B_0, A_1 \times B_1) = \alpha_1(A_0,A_1) \times \alpha_2(B_0,B_1)$

where A_0, B_0 are the minimal p-open sets of T_1 and T_2 respectively and $A_1 \times B_2 \in P^{X \times Y}$.

$= \alpha_1(A_1, A_0) \times \alpha_2(B_1,B_0)$

$= \alpha^*(A_1 \times B_1, A_0 \times B_0)$

$\therefore\ \alpha^*$ is a p-a structure on $X \times Y$.

$\therefore\ (X \times Y, T^*, \alpha^*)$ is a p-a space.

Remarks 3.4. α^* is a called the Product p-a structure and $(X \times Y, T^*, \alpha^*)$ is called the Product p-a space.

Example 3.5. Let (X,T_1,α_1) and (Y,T_2,α_2) be two p-a spaces where (X,T_1) and (Y,T_2) are two p-topological spaces.

Let $\alpha_1(A_1,B_1) = A_1 \cup B_1$ and $\alpha_2(A_2,B_2) = A_2 \cup B_2$ where $A_1,B_1 \in T_1$ and $A_2,B_2 \in T_2$.

Let $T^* = \{A_1 \times A_2 : A_1 \in T_1, A_2 \in T_2\}$ be the Product p-topology on $X \times Y$.

Let $\alpha^* : P^{X \times Y} \times P^{X \times Y} \to P^{X \times Y}$ be such that

$\alpha^*(A_1 \times A_2, B_1 \times B_2) = \alpha_1(A_1,B_1) \times \alpha_2(A_2,B_2)$

$= (A_1 \cup B_1) \times (A_2 \cup B_2)$

Then $(X \times Y, T^*, \alpha^*)$ is the Product p-a space.

Example 3.6. Let (G_1,T_1,α_1) and $G_2,T_2,\alpha_2)$ be two p-a spaces where G_1 and G_2 are any two groups and T_1, T_2 are usual p-topologies on G_1 and G_2 and α_1,α_2 are usual p-a structures on G_1 and G_2 respectively.

Let $T^* = \{ A_1 \times A_2 : A_1 \in T_1, A_2 \in T_2\}$ be the

Product p-topology on $X \times Y$.

Let $\alpha^* : P^{G_1 \times G_2} \times P^{G_1 \times G_2} \rightarrow P^{G_1 \times G_2}$ be such that

$\alpha^*(A_1 \times A_2, B_1 \times B_2) = \alpha_1(A_1,B_1) \times \alpha_2(A_2,B_2)$

$= (A_1B_1) \times (A_2B_2)$

Then $(G_1 \times G_2, T^*,\alpha^*)$ is the Product p-a space.

4. SUB p-a SPACE OF PRODUCT p-a SPACE

Proposition 4.1. Let (X,T_1,α_1) and (Y,T_2,α_2) be two p-a spaces where (X,T_1) and (Y,T_2) are p-topological spaces and α_1,α_2 are usual p-a structures on X,Y respectively.

Let $\overline{X} = X \times B_0, B_0$ is the minimal p-open set in T_2

$\overline{T} = \{A \times B_0 : A \in T_1\}$

and $\overline{\alpha} = (A \times B_0, B \times B_0) = \alpha_1(A,B) \times B_0$

Then $(\overline{X},\overline{T}_1,\overline{\alpha}_1)$ is a sub p-a space of $(X \times Y, T^*, \alpha^*)$ where T^* and α^* are the Product p-topology and Product p-a structure respectively on $X \times Y$.

Proof. First we Show that $\overline{T}_1$ is a p-topology on $\overline{X}$.

(i) $\overline{X} = X \times B_0 \in \overline{T}_1$ since $X \in \overline{T}_1$, B_0 is the minimal p-open set in T_2.

(ii) Let A_0 be the minimal p-open set in T_1. Then $A_0 \times B_0$ is the minimal p-open set in $\overline{T}_1$.

(iii) Let $A_1 \times B_0$, $A_2 \times B_0$ be any two elements of $\overline{T}_1$ where $A_1,A_2 \in T_1$

Then $(A_1 \times B_0) \cap (A_2 \times B_0) = (A_1 \cap A_2) \times B_0 \in \overline{T}_1$

since $A_1, A_2 \in T_1 \Rightarrow A_1 \cap A_2 \in T_1$

$\therefore$ $\overline{T}_1$ is a p-topology on $\overline{X}$.

Now we show that $(\overline{X},\overline{T}_1,\overline{\alpha}_1)$ is a sub p-a space of $(X \times Y, T^*, \alpha^*)$.

Let $\overline{\alpha}_1 : P^{\overline{X}} \times P^{\overline{X}} \to P^{\overline{X}}$ be such that.

(i) $\overline{\alpha}_1(A \times B_0, B \times B_0) = \alpha_1(A,B) \times B_0$, $A,B \in T_1$ and B_0 is the minimal p-open set in T_2.

(ii) $\overline{\alpha}_1(A \times B_0, B \times B_0) = \alpha_1(A,B) \times B_0$

$$= \alpha_1(B,A) \times B_0$$

$$= \overline{\alpha}_1(B \times B_0, A \times B_0)$$

$$\therefore \overline{\alpha}_1(A \times B_0, B \times B_0) \in \overline{T}_1$$

(iii) $\overline{\alpha}_1(A_0 \times B_0, A \times B_0) = \alpha_1(A_0,A) \times B_0$ where A_0 is the minimal p-open set in T_1.

$$= \alpha_1(A,A_0) \times B_0$$

$$= \overline{\alpha}_1(A \times B_0, A_0 \times B_0)$$

$\therefore$ $(\overline{X},\overline{T}_1,\overline{\alpha}_1)$ is a sub p-a space of $(X \times Y, T^*, \alpha^*)$.

Proposition 4.2. Let (X,T_1,α_1) and (Y,T_2,α_2) be two p-a spaces where (X, T_1) and (Y,T_2) are p-topological spaces, α_1 and α_2 are p-a structures on X and Y respectively. Let $(\overline{X},\overline{T}_1,\overline{\alpha}_1)$ be a sub p-a space of $(X \times Y, T^*, \alpha^*)$ where T^* and α^* are the product p-topology and Product p-a structure respectively on $X \times Y$.

Let $f : (X.T_1,\alpha_1) \to (\overline{X},\overline{T}_1,\overline{\alpha}_1)$ be an onto mapping such that

$f(\{x\}) = \{x\} \times B_0$ where B_0 is the minimal p-open set in T_2

Then f is a p-a homomorphism.

Proof. (i) $f(x) = X \times B_0 \in \overline{T}_1$ whenever $X \in T_1$

$\therefore$ f is p-open.

$f^{-1}(\overline{X}) = f^{-1}(X \times B_0) = X \in T_1$ whenever $\overline{X} \in \overline{T}_1$

$\therefore$ f is p-continuous.

$\therefore$ f is both p-open and p-continuous.

(ii) $f(\alpha_1(A,B)) = \alpha_1(A,B) \times B_0$, $A,B \in T_1$

$$= \overline{\alpha}_1(A \times B_0, B \times B_0)$$

$$= \overline{\alpha}_1(f(A), f(B))$$

$$\text{and} \quad \alpha_1(f^{-1}(\overline{A}), f^{-1}(\overline{B})) = \alpha_1(f^{-1}(A \times B_0), f^{-1}(B \times B_0))$$
$$= \alpha_1(A,B)$$
$$= f^{-1}(\alpha_1(A,B) \times B_0)$$
$$= f^{-1}(\overline{\alpha}_1(A \times B_0, B \times B_0))$$
$$= f^{-1}(\overline{\alpha}_1(\overline{A}, \overline{B}))$$

$\therefore$ f is a p-a homomorphism.

Remarks 4.3. $(\overline{X}, \overline{T}_1, \overline{\alpha}_1)$ may be identified by (X,T_1,α_1).

(X,T_1,α_1) is called the projection of $(X \times Y, T^*, \alpha^*)$ by (Y,T_2,α_2)

Similarly we may speak about the projection

(Y,T_2,α_2) of $(X \times Y, T^*, \alpha^*)$ by (X,T_1,α_1).

ACKNOWLEDGEMENT

I offer my heartiest thanks to Prof. Dr. N. R. Das, Head of the Department of Mathematics, Gauhati University, for his valuable advices in preparing this paper.

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Jour. Pure Math., Vol. 19, 2002, pp. 81-85

WEAK FINITE OPERATORS

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ABSTRACT : Let $L(H)$ denote the algebra of all bounded linear operators on a separable infinite dimensional complex Hilbert space H into itself. For $A \in L(H)$, we define the derivation $\delta_A : L(H) \to L(H)$ by $\delta_A(X) = AX - XA$. In this paper we introduce the notion of weak finite operators for which we give characterization and we prove that this class of operators is norm dense in $L(H)$ by generalizing H. Yang and S. N. El alami's results [8], [25].

2000 Mathematics Subject Classification : 47A30, 47B10, 47B47.

Key words : Derivation, Weak finite operators.

1. INTRODUCTION

Let $L(H)$ be the algebra of all bounded linear operators acting on a complex separable Hilbert space H. The generalized derivation operator $\delta_{A,B}$ associated with (A,B), defined on $L(H)$ by

$$\delta_{A,B}(X) = AX - XB$$

was initially studied by M. Rosenblum [13]. The properties of such operators have been much studied (see for example [1], [2], [4], [12], [13] and [14]).

If $A = B$, $\delta_{A,A} = \delta_A : L(H) \to L(H)$ is called the inner derivation defined by

$$\delta_A(X) = AX - XA.$$

The theory of derivations has been extensively dealt with in the literature (see, [5], [8], [9], [11], [16], [17], [18] and [19]. Let $\mathcal{N}$ be the set $\left\{A \in L(H) : I \notin \overline{R(\delta_A)}\right\}$. In [15] we gave some operators for which $I \notin \overline{R(\delta_A)}$. In [25] H. Yang, shows that the set $\mathcal{N}$ is norm-dense in $L(H)$. Also S. N. Elalami [8], shows that the set $\mathcal{M}_w = \left\{A \in L(H) : I \notin \overline{R(\delta_A)}^w\right\}$, is norm-dense in $L(H)$, where $\overline{R(\delta_A)}^w$ denotes the weak closure of the range of δ_A. In order

Supported by the Research Centre Project No. **Math/1422/10**

to generalize these results we prove that the set $\mathcal{T}_w = \left\{A \in L(H) : \forall K \in \mathcal{K}(H), I \notin \overline{R(\delta_A)}^w\right\}$ is also norm-dense in $L(H)$, where $\mathcal{K}(H)$ is the ideal of all compact operators.

Definition 1.1. *We shall say that a certain property (P) (of operators acting on a Hilbert space) is a bad property, written b-property.*

(i) If A has the Property (P) and T similar to A, then $\alpha + \beta A$ has the property (P), for all $\alpha \in \mathbb{C}$ and $\beta \neq 0$,

(ii) If A has the property (P) and T similar to A, then T has the property (P), and

(iii) If A has the property (P) and $\sigma(A) \cap \sigma(B) \neq \phi$, then $A \oplus B$ has the property (P).

Example of bad properties are frequent in the literature ; namely, (1) T is not cyclic, (2) the spectrum of T is disconnected (or $\sigma(T)$) has infinitely many components, or c components, where c is the power of the continuim), (3) $\sigma(T)$ has non empty interior, (4) the commutant of T is not algebraic, etc., are examples of properties satisfying (i), (ii), (iii).

Definition 1.2. *An operator $A \in L(H)$ is called weak finite, if $I \notin \overline{R(\delta_A)}^w$.*

Theorem 1.3. *The set $\mathcal{T}_w = \left\{A \in L(H) : \forall K \in \mathcal{K}(H), I + K \notin \overline{R(\delta_A)}^w\right\}$ is norm-dense in $L(H)$.*

Proof. By using [10], Theorem 3.5.1, it suffices to prove that the property $A \in \mathcal{T}w$ is a b-property. It is easy to see that $R(\delta_A) = R(\delta_{\alpha A+\beta})$ for $\alpha \in \mathbb{C}$, $\alpha \neq 0$, $\beta \in \mathbb{C}$ and $X \in L(H)$; hence if $A \in \mathcal{T}_w$, then $\alpha A + \beta \in \mathcal{T}_w$. Now if $S \in L(H)$ and S is invertible, then for all $X \in L(H)$,

$$S(AX - XA)S^{-1} = (SAS^{-1})(SXS^{-1}) - (SXS^{-1})(SAS^{-1}).$$

Thus $S\overline{R(\delta_A)}^w S^{-1} = \overline{R(\delta_{SAS^{-1}})}^w$. Hence if $I + K \in \overline{R(\delta_A)}^w$, then $I + SKS^{-1} \in \overline{R(\delta_{SAS^{-1}})}^w$.

It follows by the above argument that if $\overline{R(\delta_A)}^w$ contains $I + K$, then it is also true for all operator similar to A. Hence $A \in \mathcal{T}_w$ is invariant for similarity.

Let $\varepsilon = H \oplus H$ and $B = A \oplus C$. Suppose that there exists $\{X_\alpha\} \in L(\varepsilon)$ such that $[(A \oplus C) X_\alpha - X_\alpha(A \oplus C) \xrightarrow{w} I_\varepsilon \oplus K$. Let P_0 be the orthogonal projection on H, K_1 denotes the compression of K to H, i.e., $K_1 = P_0KP_0 | H$ and X_{α_1} denotes the compression of X_α to H. Then $AX_{\alpha_1} - X_{\alpha_1}A \xrightarrow{w} I_H \oplus K_1$. So, if $A \oplus C \notin \mathcal{T}_w$ then $A \notin \mathcal{T}_w$. Note that the hypothesis on the spectrum of C can be dropped here. ■

2. EXAMPLES OF WEAK FINITE OPERATORS

Theorem 2.1. *Every operator $A \in L(H)$ which has a pole of order v, is weak finite.*

Proof. The operator A can be decomposed as $A = B \oplus C$ on $H = R(P_\lambda) \oplus G$, where $(B-\lambda)^v = 0$ on $R(P_\lambda)$ and P_λ is the Riesz projection. Since B is algebraic, $I \notin \overline{R(\delta_B)}^w$. Hence $I \notin \overline{R(\delta_A)}^w$. ■

Lemma 2.2. *If $H = \oplus_{i=1}^n H_i$ where $\dim H_n < \infty$ (orthogonal direct sum), and if $A = \oplus_{i=1}^n A_i$ on H, then every operator in $\overline{R(\delta_A)}^w$ vanishes. Consequently*

$$\overline{R(\delta_A)}^w \cap \{A^*\}' = \{0\}.$$

Proof. *Suppose that*

$$AX_\alpha - X_\alpha A \xrightarrow{w} P,$$

where P is a positive operator and $\{X_\alpha\}$ a generalized sequence in $L(H)$. Noting by Q_n the orthogonal projection on H_n, we get

$$AX_\alpha^{(n)} - X_\alpha^{(n)} A \xrightarrow{w} Q_n(P \mid H_n),$$

where $X_\alpha^{(n)}$ is the compression of X_α to H_n. *Since $\dim H_n < \infty$, $Q_n(P \mid H_n) \in R(\delta_{A_n})$ this implies that $Q_n(P \mid H_n) = 0$ since this last operator is positive on H_n. Then*

$$0 = Q_n P Q_n = \left(\sqrt{P} Q_n\right) * \left(\sqrt{P} Q_n\right)$$

from where $\sqrt{P} Q_n = 0$ or $PQ_n = 0$. As $\sum_{i=1}^n Q_i = I$, it follows that $P = 0$.

For the consequence, it suffices to remark that if

$$T \in \overline{R(\delta_A)}^w \cap \{A^*\}',$$

then $T * T \in \overline{R(\delta_A)}^w$. ■

Lemma 2.3. *Let $A \in L(H)$. If A is normal and countable, then every positive operator in $\overline{R(\delta_A)}^w$ vanishes. Consequently*

$$\overline{R(\delta_A)}^w \cap \{A\}' = \{0\}.$$

Proof. *Let T be a positive operator. In the decomposition of*

$H = \overline{R(A)} \oplus \ker A$,

We can write

$$A=\begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix};\ T=\begin{bmatrix} R & * \\ * & S \end{bmatrix}.$$

If $T \in \overline{R(\delta_A)}^w$, *then* $S = 0$ *and* $R \in \overline{R(\delta_A)}^w$. *Since* $\overline{R(A)}$ *is a countable orthogonal basis composed of eingenvectors of A. Also* $R = 0$ *by Lemma 2.2. Since T is positive, it results that* $T = 0$.

Concerning the consequence it suffieces to remark that $\{A\}' = \{A^*\}'$.

Theorem 2.4. *Let* $A \in L(H)$. *If A has a countable spectrum and if P(A) is normal for certain non-constant polynomial P, then A is weak finite.*

Proof. It follows from [15, Lemma 3] that we can assume A has no poles. Now suppose that

$$AX_\alpha - X_\alpha A \xrightarrow{w} I.$$

It is easy to see that

$$P(A)X_\alpha - X_\alpha P(A) \xrightarrow{w} P'(A),$$

which implies that

$$P'(A) \in \overline{R(\delta_{P(A)})}^w \cap \{P(A)\}'.$$

It follows from Lemma 2.3 that $P'(A) = 0$ and by the theorem of the minimum equation [7], P' vanishes on some neighborhood of $\sigma(A)$. Hence P is constant, which is absurd. ∎

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Jour. Pure Math., Vol. 19, 2002, pp. 87-95

COMMON FIXED POINTS FOR FUZZY MAPPINGS WITH THEIR ASSOCIATED MULTIMAPPINGS

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ABSTRACT : In this paper we obtain a result on fixed points of fuzzy mappings with their associated multimappings which extends the result of [15] and [3].

1. INTRODUCTION

Several fixed point theorems for fuzzy mappings have been obtained by researchers [1, 2, 4-7, 9-14]. Heilpern [7] obtained a fixed point theorem for contractive type fuzzy mappings in metric space. Also Lee and Cho [9] studied fixed point theorems for contractive type fuzzy mappings which are fuzzy analogue of fixed point theorems for contractive type multivalued mappings (see [8]). Lee et al. [10-13] discussed common fixed points of a sequence of fuzzy mappings, especially they [13] showed existence of common fixed points for a pair of fuzzy mappings.

2. PRELIMINARIES

We state some useful notations, definitions and results.

Definition 2.1. Let (X,d) be any metric linear space. A fuzzy set in X is a function with domain X and values in [0,1].

Definition 2.2. If A is a fuzzy set and $x \in X$, the function values A(x) (or $\mu A(x)$ is called the grade of membership of x in A.

Definition 2.3. The α-level set of a fuzzy set A, denoted by

$A_\alpha = \{x : A(x) \geq \alpha\}$ if $\alpha \in (0,1)$ and $A_0 = \{x : A(x) > 0\}$.

Definition 2.4. A fuzzy set A is said to be an approximate quantity iff A_α is compact and convex in X for each $\alpha \in [0,1]$ and $\sup_{x \in X} A(x) = 1$.

We denote by W(X) the sub collection of approximate quantities, C(X) the set of compact subsets of X, (C(X),H) the Hausdorff metric space and D(A,B) = $\inf_{x \in A,\, y \in B} d(x,y)$.

Definition 2.5. Let X be an arbitrary set and Y be any metric linear space. F is said to be a fuzzy mapping iff F is a mapping from the set X into W(Y) i.e. F(x) $\in$ W(Y) for each x $\in$ X.

Definition 2.6. [16] A point p $\in$ X is called a fixed point of a fuzy mapping F : X $\to$ W(X) if $F_p(P) \geq F_p(x)$, for all x $\in$ X.

Definition 2.7. [16] If F : X $\to$ W(X) be a fuzzy mapping. Then, an associated multi mapping $F^\wedge : X \to CB(X)$ is defined by

$$F^\wedge(x) = \{y \in X : F_x(y) = \max_{u \in X} F_x(u)\}$$

Definition 2.8. [7] $D_\alpha(A,B) = \inf_{x \in A_\alpha\ y \in B_\alpha} d(x,y)$

$$H_\alpha(A,B) = \underset{\alpha}{dist}(A_\alpha, B_\alpha);$$

$$D(A,B) = \sup_\alpha H_\alpha(A,B);$$

The following Lemma is due to Heilpern.

Lemma 2.9. [7] If $\{x_0\} \subset A$, then $D_\alpha(x_0,B) \leq H_\alpha(A,B)$ for each B $\in$ W(X).

Lemma 2.10. $F_p^\wedge(p) \geq F_p^\wedge(x)$ iff $p \in F^\wedge(p)$ for all x $\in$ X.

Ray [15] proved the following.

Theorem 2.11. Let (X,d) be a complete metric space, R^+ the set of all non negative real numbers and w : $R^+ \to R^+$ is a continuous function such that $0 < w(r) < r$, for all r in $R^+ - \{0\}$. Then self mappings f, g and h of X have a unique common fixed point if

(i) $d(fx,gy) \leq d(hx,hy) - w(d(hx,hy))$,

(ii) h is continuous,

(iii) $f(X) \cup g(X) \subseteq h(X)$.

Change [3] proved the following theorem.

Theorem 2.12. Let F, G : X $\to$ W(X) be two fuzzy mappings and $F^\wedge, G^\wedge$ be their associated multimappings respectively. Suppose that for any x,y $\in$ X, the following holds

$$H(F^\wedge(x), G^\wedge(y)) \leq \phi(d(x,y), d(x,F^\wedge(x)), d(x,G^\wedge(y)), d(y,F^\wedge(y)\})$$

where the function ϕ satisfies the following conditions

(i) $\phi : [0,\infty)^5 \to [0,\infty)$ is non decreasing for each variable and ϕ is upper semi continuous,

(ii) ϕ (t,t,t,at,bt) $\leq \phi$ (t), for all $t \geq 0$, where ϕ (t) is a function from $[0,\infty)^5$ into $[0,\infty)$ such that $\phi(t) < t$ for all $t > 0$, $\phi(0) = 0$; a, b = 0, 1, 2; a + b =2.

Let $\beta > 1$, $x_0 \in X$, $x_1 \in F^\wedge$ (x_0) and define a non-negative real sequence as following ;

$t_{k+1} = t_k + \phi(\beta(t_k - t_{k-1}))$; k = 1, 2, ...; $t_0 = 0$; $t_1 > d(x_0,x_1)$.

If $(t_k\}$ converges, then $F^\wedge$ and $G^\wedge$ have a unique common fixed point.

3. MAIN RESULT

We prove the following.

Theorem 3.1. Let (X,d) be a complete metric space. Let F,G : X $\rightarrow$ W(X) be two fuzzy mappings and $F^\wedge$, $G^\wedge$ be their associated multimappings defined from X into C(X) (the set of compact subsets of X) satisfying

$$\text{(i)}\quad H^2(F^\wedge x,G^\wedge Y) \leq \max\{D^2(x,F^\wedge x),D^2(y,G^\wedge y),D(x,F^\wedge x)D(y,G^\wedge y)$$
$$(\tfrac{1}{2})d(x,y)[D(x,G^\wedge y)+D(y,F^\wedge x)]\}$$
$$-w(\max\{D^2(x,F^\wedge x),D^2(y,G^\wedge y),D(x,F^\wedge x)D(y,G^\wedge y),$$
$$(\tfrac{1}{2})d(x,y)[D(x,G^\wedge y)+D(y,F^\wedge x)]\}$$

for all x,y $\in$ X ; w : $R^+ \rightarrow R^+$ a non-decreasing continuous function such that $0 < w(r) < r$, for all $r > 0$ and $w(0) = 0$.

Then there exists a common fixed point of $F^\wedge$ and $G^\wedge$. Also F and G have a common fixed point.

Proof. Let $x_0 \in X$. Since C(X) is compact. So we can construct a sequence

$\{x_n\}$ such that $x_{2n+1} \in F^\wedge x_{2n}$ and $x_{2n+2} \in G^\wedge x_{2n+1}$ $\forall$n= 0,1,2 ... with

$d(x_{2n},x_{2n-1}) = D(x_{2n},F^\wedge x_{2n})$ and $d(x_{2n+1},x_{2n+2}) = D(x_{2n+1},G^\wedge x_{2n+1})$ and

$d(x_{2n+1},x_{2n+2}) \leq H(F^\wedge x_{2n},G^\wedge x_{2n+1})$

Using inequality (i), we have

$$d^2(x_1,x_2) \leq H^2(F^\wedge x_0,G^\wedge x_1)$$
$$\leq \max\{D^2(x_0,F^\wedge x_0),D^2(x_1,G^\wedge x_1),D(x_0,F^\wedge x_0)D(x_1,G^\wedge x_1),$$
$$(\tfrac{1}{2})d(x_0,x_1)[D(x_0,G^\wedge x_1)+D(x_1,F^\wedge x_0)]\}$$

$$-w(\max\{D^2(x_0,F^\wedge x_0),D^2(x_1,G^\wedge x_1),D(x_0,F^\wedge x_0)D(x_1,G^\wedge x_1),$$
$$(½)d(x_0,x_1)[D(x_0,G^\wedge x_1)+D(x_1,F^\wedge x_0)]\}$$
$$=\max\{d^2(x_0,x_1),d^2(x_1,x_2),d(x_0,x_1)d(x_1,x_2),(½)d(x_0,x_1)d(x_0,x_2)\}$$
$$-w(\max\{d^2(x_0,x_1),d^2(x_1,x_2),d(x_0,x_1)d(x_1,x_2),(½)d(x_0,x_1)d(x_0,x_2)\})$$
$$\le\max\{d^2(x_0,x_1),d^2(x_1,x_2),d(x_0,x_1)d(x_1,x_2),$$
$$(½)d(x_0,x_1)[d(x_0,x_1)+d(x_1,x_2)\}$$
$$-w(\max\{d^2(x_0,x_1),d^2(x_1,x_2),d(x_0,x_1)d(x_1,x_2),$$
$$(½)d(x_0,x_1)[d(x_0,x_1)+d(x_1,x_2)]\}) \quad \ldots (1)$$

If $d(x_1,x_2) > d(x_0,x_1)$, then we have

$d^2(x_1,x_2) \le d^2(x_1,x_2) - w(d^2(x_1,x_2))$,

leading to a contradiction. Therefore, $d(x_1,x_2) \le d(x_0,x_1)$ and by (1)

$$d^2(x_1,x_2) \le d^2(x_0,x_1) - w(d^2(x_0,x_1)) \quad \ldots (2)$$

Further using (i), we have

$$d^2(x_2,x_3) \le H^2(G^\wedge x_1,F^\wedge x_2)$$
$$= H^2(F^\wedge x_2,G^\wedge x_1)$$
$$\le\max\{D^2(x_2,F^\wedge x_2),D^2(x_1,G^\wedge x_1),D(x_2,F^\wedge x_2)D(x_1,G^\wedge x_1)$$
$$(½)d(x_2,x_1)[D(x_2,G^\wedge x_1)+D(x_1,F^\wedge x_2)\}$$
$$-w(\max\{D^2(x_2,F^\wedge x_2),D^2(x_1,G^\wedge x_1),D(x_2,F^\wedge x_2)D(x_1,G^\wedge x_1)$$
$$(½)d(x_2,x_1)[D(x_2,C^\wedge x_1)+D(x_1,F^\wedge x_2)\})$$
$$=\max\{d^2(x_2,x_3),d^2(x_1,x_2),d(x_2,x_3)d(x_1,x_2),$$
$$(½)d(x_1,x_2)d(x_1,x_3)\}$$
$$-w(\max\{d^2(x_2,x_3),d^2(x_1,x_2),d(x_2,x_3)d(x_1,x_2),$$
$$(½)d(x_1,x_2)d(x_1,x_3)\}).$$

This implies with same arguments as above

$$d^2(x_2,x_3) \le d^2(x_1,x_2) - w(d^2(x_1,x_2)) \quad \ldots (3)$$

Similarly,

$d^2(x_3,x_4) \leq d^2(x_2,x_3) - w(d^2(x_2,x_3))$ ... (4)

$d^2(x_n,x_{n+1}) \leq d^2(x_{n-1},x_n) - w(d^2(x_{n-1},x_n))$ ... (3)

Adding (2) to (5), we have

$$\sum_{i=0}^{n-1} w(d^2(x_i,x_{i+1}) \leq d^2(x_0,x_1) - d^2(x_n,x_{n+1}) \leq d^2(x_0,x_1)$$

This implies, $\sum_{i=0}^{n-1} w(d^2(x_i,x_{i+1}) < \infty$ and $\lim w(d^2(x_n,x_{n+1})) = 0$ ... (6)

Since $\{d^2(x_n,x_{n+1})$ is a decreasing sequence of non-negative terms, therefore (6) implies that $\lim_{n\to\infty} d^2(x_n,x_{n+1}) = 0$ and so $\lim_{n\to\infty} d(x_n,x_{n+1}) = 0$

Suppose that $\{x_n\}$ is not a Cauchy sequence, then there is an $\varepsilon > 0$ such that for each even integer 2K, there are even integers $2m > 2n > 2K$ such that $d(x_{2m}.x_{2n}) > \varepsilon$. ... (7)

Also for each even integer 2k, we can find the least even integer 2m, exceeding 2n such that

$d(x_{2n}, x_{2m-2}) \leq \varepsilon$ ... (8)

then $\varepsilon < d(x_{2m},x_{2n}) \leq d(x_{2n},x_{2m-2}) + d(x_{2m-2},x_{2m-1}) + d(x_{2m-1}, x_{2m})$

This implies $d(x_{2m},x_{2n}) \to \varepsilon$ as $k \to \infty$.

Using triangular property of metric space, we have

$| d(x_{2m},x_{2n+1}) - d(x_{2m},x_{2n}) | \leq d(x_{2n},x_{2n+1})$,

$| d(x_{2m+1},x_{2n+1}) - d(x_{2m},x_{2n+1}) | \leq d(x_{2m},x_{2m+1})$,

$| d(x_{2m},x_{2n+2}) - d(x_{2m},x_{2n+1}) | \leq d(x_{2n+1},x_{2n+2})$,

and $| d(x_{2m},x_{2n+2}) - d(x_{2m+1},x_{2n+1}) | \leq d(x_{2n+1},x_{2n+2})$

which implies on letting lim $k \to \infty$,

$d(x_{2m},x_{2m+1}) \to \varepsilon$, $d(x_{2m+1},x_{2n+1}) \to \varepsilon$, $d(x_{2m},x_{2n+2}) \to \varepsilon$

and $d(x_{2m+1},x_{2n+2}) \to \varepsilon$.

Now using (i), we have

$$d^2(x_{2m+1},x_{2n+2}) \leq H^2(F^\wedge x_{2m}, G^\wedge x_{2n+1})$$

$$\leq \max\{D^2(x_{2m}, F^\wedge x_{2m}), D^2(x_{2n+1}, G^\wedge x_{2n-1}),$$

$$D(x_{2m}, F^\wedge x_{2m})D(x_{2n+1}, G^\wedge x_{2n+1}),$$

$$(½)d(x_{2m}, x_{2n})[D(x_{2m}, G^\wedge x_{2n+1}) + D(x_{2n+1}, F^\wedge x_{2m})]\}$$

$$-w \max\{D^2(x_{2m}, F^\wedge x_{2m}), D^2(x_{2n+1}, G^\wedge x_{2n+1}),$$

$$D(x_{2m}, F^\wedge x_{2m})D(x_{2n+1}, G^\wedge x_{2n+1}),$$

$$(½)d(x_{2m}, x_{2n})[D(x_{2m}, G^\wedge x_{2n+1}) + D(x_{2n+1}, F^\wedge x_{2m})]\}$$

or

$$d^2(x_{2m+1}, x_{2n+2}) \leq \max\{d^2(x_{2m}, x_{2m+1}), d^2(x_{2n+1}, x_{2n+2}),$$

$$d(x_{2m}, x_{2m-1})d(x_{2n+1}, x_{2n+2}),$$

$$(½)d(x_{2m}, x_{2n})[d(x_{2m}, x_{2n+2}) + d(x_{2n+1}, x_{2m+1})]\}$$

$$-w(\max\{d^2(x_{2m}, x_{2m+1}), d^2(x_{2n+1}, x_{2n+2}),$$

$$d(x_{2m}, x_{2m+1})d(x_{2n+1}, x_{2n+2}),$$

$$(½)d(x_{2m}, x_{2n})[d(x_{2m}, x_{2n+2}) + d(x_{2n+1}, x_{2m+1})]\}$$

which implies on letting $k \to \infty$,

$$\varepsilon^2 \leq \varepsilon^2 - w(\varepsilon^2)$$

which is a contradiction. Thus $\{x_n\}$ is a Cauchy sequence. Since X is complete, so $\{x_n\}$ converges to a point $u \in X$. Thus, we have

$$D^2(x_{2n+1}, G^\wedge u) \leq H^2(F^\wedge x_{2n}, G^\wedge u)$$

$$\leq \max\{D^2(x_{2n}, F^\wedge x_{2n}), D^2(u, G^\wedge u), D(x_{2n}, F^\wedge x_{2n})D(u, G^\wedge u),$$

$$(½)d(x_{2n}, u)[D(x_{2n}, G^\wedge u) + D(u, F^\wedge x_{2n})]\}$$

$$-w(\max\{D^2(x_{2n}, F^\wedge x_{2n}), D^2(u, G^\wedge u), D(x_{2n}, F^\wedge x_{2n})D(u, G^\wedge u),$$

$$(½)d(x_{2n}, u)[D(x_{2n}, G^\wedge u) + D(u, F^\wedge x_{2n})]\}$$

$$\leq \max\{d^2(x_{2n}, x_{2n+1}), D^2(u, G^\wedge u), d(x_{2n}, x_{2n+1})D(u, G^\wedge u),$$

$$(½)d(x_{2n}, u)[D(x_{2n}, G^\wedge u) + D(u, F^2 x_{2n})]\}.$$

Taking limit $n \to \infty$, we have

$$D^2(u,G^\wedge u) \le \max\{0, D^2(u,G^\wedge u),0,0\} - w(\max\{0,D^2(u,G^2u),0,0\})$$

$$D^2(u,G^\wedge u) \le D^2(u,G^2u) - w(D^2(u,G^2u))$$

which implies that $u \in G^\wedge u$.

Similarly, we can prove that $u \in F^\wedge u$. Hence $u \in F^\wedge u \cap G^\wedge u$.

Using Lemma 2.10, it is obvious that $u \in Fu \cap Gu$.

Corollary 3.2. Let $\{F_i\}$ be a sequence of fuzzy mappings of X to W (X) and $\{F^\wedge{}_i\}$ the sequence of its associated multimappings from X to C(X).

Suppose, for any positive integers i, j ; i ≠ j and x,y ∈ X, following condition holds.

$$H^2(F_i^\wedge x, F_j^\wedge y) \le \max\{D^2(x,F_i^\wedge x), D^2(y,F_j^\wedge y), D(x,F_i^\wedge x)D(y,F_j^\wedge y),$$
$$(½)d(x,y)[D(x,F_j^\wedge y)+D(y,F_i^\wedge x)]\}$$
$$-w(\max\{D^2(x,F_i^\wedge x), D^2(y,F_j^\wedge y), D(x,F_i^\wedge x)D(y,F_j^\wedge y),$$
$$(½)d(x,y)[D(x,F_j^\wedge y)+D(y,F_i^\wedge x)]\}$$

where w : $R^+ \to R^+$ is a non-decreasing continuous function such that 0 < W (r) < r, for all r > 0 and w (0) = 0. Then there exists a fixed point of $\{F_i^\wedge\}$.

Corollary 3.5. Let (X, d) be complete metric space and $\{F_i\}$, $\{G_i\}$ the sequence of fuzzy mapping of X to W(X) and $\{F_i^\wedge\},\{G_i^\wedge\}$ be the sequences of their associated multi-mapping of X to C(X) (the set of compact subsets of X) converging pointwise to the associated multimappings $F^\wedge$, $G^\wedge$ of fuzzy mapping F and G respectively, Satisfying

$$H^2(F_n^\wedge x, G_n^\wedge x) \le \max\{D^2(x,F_n^\wedge x), D^2(y,G_n^\wedge y), D(x,F_n^\wedge x)D(y,G_n^\wedge y),$$
$$(½)d(x,y)[D(x,G_n^\wedge y)+D(y,F_n^\wedge x)]\}$$
$$-w(\max\{D^2(x,F_n^\wedge x), D^2(y,G_n^\wedge y), D(x,F_n^\wedge x)D(y,G_n^\wedge y),$$
$$(½)d(x,y)[D(x,G_n^\wedge y)+D(y,F_n^\wedge x)]\})$$

Then $F^\wedge$ and $G^\wedge$ have a unique common fixed point.

Proof. Let x_0 ∈ X. Define the sequence $\{x_n\}$ with

$x_{2n+1} \in F_n^\wedge x_{2n}$ and $x_{2n+2} \in G_n^\wedge x_{2n+1}$ $\forall n=0,1,2$ ..

such that $d(x_{2n+1,}x_{2n+2}) \leq H(F_n^\wedge x_{2n,} G_n^\wedge x_{2n+1})$.

Now, for x,y $\in$ X, we have

$|D(y, F_n^\wedge x) - D(y, F^\wedge x)| \leq H(F_n^\wedge x, F^\wedge x)$;

$|D(y, G_n^\wedge y) - D(y, G^\wedge y)| \leq H(G_n^\wedge y, G^\wedge y)$

$|D(x, F_n^\wedge x) - D(x, F^\wedge x)| \leq H(F_n^\wedge x, F^\wedge x)$ and

$|D(x, G_n^\wedge y) - D(x, G^\wedge y)| \leq H(G_n^\wedge y, G^\wedge y)$.

Now, since H is continuous and $\{F_n^\wedge\}, \{G_n^\wedge\}$ cnverge pointwise to $F^\wedge$ and $G^\wedge$ respectively, hence we get

$$H^2(F^\wedge x, G^\wedge y) \leq \max\{D^2(x, F^\wedge x), D^2(y, G^\wedge y), D(x, F^\wedge x)D(y, G^\wedge y),$$

$$(½)d(x,y)[D(x, G^\wedge y) + D(y, F^\wedge x)]\}$$

$$-w(\max\{D^2(x, F^\wedge x), D^2(y, G^\wedge y), D(x, F^\wedge x)D(y, G^\wedge y),$$

$$(½)d(x,y)[D(x, G^\wedge y) + D(y, F^\wedge x)]\}$$

Now rest part of the proof is similar to the proof of the theorem 2.11.

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Jour. Pure Math., Vol. 19, 2002, pp. 97-101

$\mathcal{H}^*$ RELATION ON A π-INVERSE SEMIGROUP *

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ABSTRACT : The aim of this paper is to give a necessary and sufficient condition for the $\mathcal{H}^*$ relation to be a congruence on a π-inverse semigroup. The work of M. K. Sen on an inverse semigroup is further developed.

AMS : 20M10

Key words : π-inverse semigroup, GV-inverse semigroup, $\mathcal{H}^*$ relation, Congruence.

1. INTRODUCTION

In [1], generalized Green's relations $\mathcal{L}^*$, $\mathcal{R}^*$, $\mathcal{H}^*$, $\mathcal{T}$ on π-regular semigroups are defined. It is noted that $\mathcal{H}^*$ relation is a congruence on quasi-clifford semigroups and GV-inverse semigroups. However, $\mathcal{H}^*$ relation is not necessary a congruence on π-orthodox semigroups, π-inverse semigroups, strong π-inverse semigroups, and even GV-semigroups. We will give a necessary and sufficient condition for the $\mathcal{H}^*$ relation to be a congruence on π-inverse semigroups. As a corollary, the result of [3] is given.

An element a of semigroup S is called π-regular if there exists a positive integer m such that $a^m \in a^m S a^m$. The semigroup is called π-regular if all its elements are π-regular. In particular, if a π-regular semigroup contains only one idemodent, then it is called a π-group. By a π-inverse semigroup, we mean a π-regular semigroup in which every regular element has an unique inverse element. Call a π-inverse semigroup stong if its set of idempotents forms a subsemigroup. A semigroup is called a GV-semigroup if it is π-regular and all its regular elements are completely regular. Call a GV-semigroup GV-inverse if it is π-inverse. By a quasi-clifford semigroup, we mean a GV-inverse semigroup whose set of idempotents forms a semilattice.

Throughtout this paper, E(S) is the set of all idempotents of the semigroup S, and Reg S is the set of all regular elements of the semigroup S. Also, we write $a^n = r(a)$, where

* This work is supported by Shan Dong Natural Science Fundation (Y99A03) and Shan Dong Education Committee Foundation (J01P01).

a is an arbitrary element of the π-regular semigroup S and n is the smallest positive integer such that $a^n \in Reg\ S$. For the sake of convenience, we just call n the smallest regular power of a.

In [1], *-Green's relations on the π-regular semigroup S are introduced :

$$a\mathcal{L}^*b \Leftrightarrow Sr(a) = Sr(b);$$

$$a\mathcal{R}^*b \Leftrightarrow r(a)S = r(b)S;$$

$$a\mathcal{T}^*b \Leftrightarrow Sr(a)S = Sr(b)S;$$

$$\mathcal{H}^* = \mathcal{L}^* \cap \mathcal{R}^*.$$

Throughout this paper, $\mathcal{L}^*$, $\mathcal{R}^*$ *and* $\mathcal{H}^*$ are *-Green's relations on the semigroup S. For other notations and terminologies not given in this paper, the reader is refered to the texts of J. M. Howie [2] and Bogdanovic [1].

2. PRELIMINARIES

We give here some descriptions for the π-inverse semigroups, the GV-inverse semigroups and the $\mathcal{H}^*$ relations on the π-inverse semigroups.

Lemma 2.1. [1] Let S be a π-regular semigroup. Then

1. H_e^* is a π-group, for all $e \in E(S)$, where H_e^* is the $\mathcal{H}^*$-class containing e of S;
2. $a\mathcal{H}^*b$ if and only if there exists $a' \in V(r(a))$ and $b' \in V(r(b))$ such that $r(a)a' = r(b)b'$ and $a'r(a) = b'r(b)$, for all $a, b \in S$;
3. If S is a π-group, then the group Reg$S$ is an ideal of S;
4. S is π-inverse if and only if there exists a positive integer n such that $(ef)^n = (fe)^n \in E(S)$, for all $e, f \in E(S)$;
5. S is strong π-inverse if and only if $E(S)$ is a semilattice; if and only if Reg$S$ is an inverse subsemigroup;
6. S is a quasi-clifford semigroup if and only if Reg$S$ is a clifford subsemigroup of S.

Lemma 2.2. [1] Let S be a π-regular semigroup. Then the following conditions are equivalent :

1. S is GV-inverse;
2. S is a semilattice of π-groups;

3. S is a GV-semigroup and there exists a positive integer n such that $(ef)^n = (fe)^n$, for all e, f $E(S)$;

4. $\mathcal{H}^* = \mathcal{T}^*$ is a semilattice congruence on S.

Lemma 2.3. Let S be a π-inverse semigroup and $a\mathcal{H}^*b$ where $a, b \in$ S. Then $r(a)r(b)^{-1}$, $r(b)r(a)^{-1}$, $r(a)^{-1}r(b)$, and $r(b)^{-1}r(a) \in \bigcup_{e\in E(s)} H_e^*$

Proof. Let n be the smallest regular power of $r(a)r(b)^{-1}$. Then, by the definition of $\mathcal{H}^*$, we have $r(a)r(b)^{-1}\mathcal{H}^*(r(a)r(b)^{-1})^n$.We show here $(r(a)r(b)^{-1})^n\mathcal{H}^*r(b)r(b)^{-1}$. Noting that S is a π-inverse semigroup and using (2) of Lemma 2.1, we can show

$$\begin{aligned}
Sr(a)r(b)^{-1} &= Sr(b)r(b)^{-1}r(b)r(b)^{-1}\\
&= Sr(b)r(a)^{-1}r(a)r(b)^{-1}\\
&\subseteq Sr(a)r(b)^{-1}\\
&= Sr(a)r(a)^{-1}r(a)r(b)^{-1}\\
&= Sr(b)r(b)^{-1}r(a)r(b)^{-1}\\
&= Sr(b)r(b)^{-1}r(b)r(b)^{-1}r(a)r(b)^{-1}\\
&= Sr(b)r(a)^{-1}r(a)r(b)^{-1}r(a)r(b)^{-1}\\
&= Sr(b)r(a)^{-1}(r(a)r(b)^{-1})^2\\
&= Sr(b)r(a)^{-1}r(b)r(b)^{-1}(r(a)r(\mathrm{b})^{-1})^2\\
&\subseteq Sr(b)r(b)^{-1}(r(a)r(b)^{-1})^2\\
&= Sr(b)r(a)^{-1}(r(a)r(b)^{-1})^3\\
&\subseteq Sr(b)r(b)^{-1}(r(a)r(b)^{-1})^3\\
&\subseteq \cdots\\
&\subseteq S(r(a)r(b)^{-1})^n.
\end{aligned}$$

Also,

$$\begin{aligned}
S(r(a)r(b)^{-1})^n &= S(r(a)r(b)^n r(b)r(b)^{-1}\\
&\subseteq Sr(b)r(b)^{-1}.
\end{aligned}$$

Hence,

$$(r(a)r(b)^{-1})^n \mathcal{L}^* r(b)r(b)^{-1}.$$

Similarly,

$$(r(a)r(b)^{-1})^n \mathcal{R}^* r(b)r(b)^{-1}.$$

As a consequence, we derive that

$$(r(a)r(b)^{-1})\,\mathcal{H}^*(r(a)r(b)^{-1}\mathcal{H}^*r(b)r(b)^{-1}.$$

This leads to $r(a)r(b)^{-1} \in \bigcup_{e\in E(S)} H_e^*$. Similarly, we can show that $r(b)r(a)^{-1}, r(a)^{-1}r(b)$ and $r(b)^{-1}r(a) \in \bigcup_{e\in E(S)} H_e^*$. The proof is completed.

Lemma 2.4. Let S be a π-inverse semigroup and $\mathcal{H}^*$ is a congruence on S. Then $\bigcup_{e\in E(S)} H_e^*$ is a GV-inverse subsemigroup of S.

Proof. Suppose that $a \in H_e^*$ and $b \in H_f^*$ where $e, f \in E(S)$. Let n be a positive integer such that $(ef)^n = (fe)^n \in E(S)$. Because $\mathcal{H}^*$ is a congruence on S, we have $ab\mathcal{H}^*ef$ and $(ab)^n\mathcal{H}^*(ef)^n = (fe)^n\mathcal{H}^*(ba)^n$. If we let m be the smallest regular power of ab and k be a positive integer such that $m^k \geq n$, then

$$ab\mathcal{H}^*(ab)^m\mathcal{H}^*(ab)^{m^2}\mathcal{H}^*\ldots\mathcal{H}^*(ab)^{m^k}\mathcal{H}^*(ef)^{m^k}\mathcal{H}^*(ef)^n.$$

Hence $ab \in \bigcup_{g\in E(S)} H_g^*$ since $(ef)^n \in E(S)$. Thus, by $a\mathcal{H}^*r(a)$, $\bigcup_{e\in E(S)} H_e^*$ is a π-regular subsemigroup of S. Similarly, $ba\mathcal{H}^*(fe)^n$. Consequently, $ab\mathcal{H}^*ba$. This means that $\mathcal{H}^*$ is a semilattice congruence on $\bigcup_{e\in E(S)} H_e^*$. Hence, by Lemma 2.1 and Lemma 2.2, we finally obtain $\bigcup_{e\in E(S)} H_e^*$ is a GV-inverse subsemigroup of S.

3. MAIN RESULTS

Theorem 3.1. Let S be a π-inverse semigroup. Then $\mathcal{H}^*$ relation is a congruence on S if and only if $T = \bigcup_{e\in E(S)} H_e^*$ is a GV-inverse subsemigroup of S and $r(ab)\mathcal{H}^*r(a)r(b)$ for all $a,b \in S$

Proof. By the above lemmas, we only need to prove the sufficiency.

Suppose now T is a GV-inverse subsemigroup of S and $r(ab)\mathcal{H}^*r(a)r(b)$ for all $a,b \in S$. Let $a\mathcal{H}^*b$, then by Lemma 2.3, we have $r(a)r(b)^{-1}, r(b)r(a)^{-1}$, $r(a)^{-1}r(b)$ *and* $r(b)r(a)^{-1} \in T$. If we write $\mathcal{H}^{*^T}$ for the $\mathcal{H}^*$ relation on T, then since T is GV-inverse semigroup, $\mathcal{H}^{*^T}$ is a semilattice congruence on T. Suppose that $c \in S$, using the above facts, we immediately have

$r(c)^{-1}r(c)r(a)r(b)^{-1}\mathcal{H}^{*T}\, r(a)r(b)^{-1}r(c)^{-1}r(c)$

$\mathcal{H}^{*T}\, r(a)r(b)^{-1}r(b)r(b)^{-1}r(c)^{-1}r(c)\mathcal{H}^{*T}\, r(a)r(b)^{-1}r(c)^{-1}r(c)r(b)r(b)^{-1}$

Hence, there exists $x \in T$ such that $r(c)^{-1}r(c)r(a)r(b)^{-1} = xr(a)r(b)^{-1}r(c)^{-1}r(c)r(b)r(b)^{-1}$. By noting that $r(b)^{-1}r(b) = r(a)^{-1}r(a)$, we can show

$r(c)r(b) = r(c)r(c)^{-1}r(c)r(a)r(b)^{-1}r(b) = [r(c)xr(a)r(b)^{-1}r(c)^{-1}]r(c)r(b)$

Similarly, there exists $y \in T$ such that

$r(c)r(b) = [r(c)yr(a)r(b)r(a)^{-1}]r(c)r(a)$.

Consequently, we derive that $r(c)r(a)\mathcal{L}^*r(c)r(b)$. Also it is obvious that $r(c)r(a)\mathcal{R}^*r(c)r(b)$.

Hence $r(c)r(a)\mathcal{H}^*r(c)r(b)$. Thereby, $ca\mathcal{H}^*r(ca)\mathcal{H}^*r(c)r(a)\mathcal{H}^*r(c)r(b)\mathcal{H}^*r(cb)\mathcal{H}^*cb$. This shows the $\mathcal{H}^*$ relation on S is a left congruence on S. In the same way, we can obtain $ac\,\mathcal{H}^*bc$, that is $\mathcal{H}^*$ is right compatible. Thus, $\mathcal{H}^*$ is indeed a congruence on S. The proof is completed.

Corollary 3.2. Let S be a strong π-inverse semigroup. Then $\mathcal{H}^*$ is a congruence on S if and only if $T = \bigcup_{e\in E(s)} H_e^*$ is a quasi-clifford subsemigroup of S and $r(ab)\mathcal{H}^*r(a)r(b)$ for all $a,b \in S$.

Proof. By noting that a GV-inverse semigroup whose set of all idempotents forms a semilattice is a quasi-clifford semigroup, the proof is completed.

Corollary 3.3 [3] Let S be a inverse semigroup, then $\mathcal{H}^*$ is a congruence on S if and only if $\bigcup_{e\in E(s)} H_e^*$ is a clifford subsemigroup of S.

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JOURNAL OF PURE MATHEMATICS UNIVERSITY OF CALCUTTA

VOLUME 19, 2002

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Price : 50/-

Printed & Published by Sri Pradip Kumar Ghosh, Superintendent, Calcutta University Press 48, Hazra Road, Kolkata — 700 019